Total No. of Questions :8]

SEAT No. :

P2710

[Total No. of Pages :3

[5528]-101

M.A./ M.Sc.

MATHEMATICS

MT-501: Real Analysis

(2013 Pattern) (Semester - I) (Credit System)

Time : 3Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Define exterior measure of a set $E \subseteq \mathbb{R}^d$. Show that the exterior measure of rectangle $\mathbb{R}$ is equal to its volume. **[5]**

b) If $E \subseteq \mathbb{R}^d$ then prove that $m_*(E) = \inf m_*(\vartheta)$ where the infimum is taken over all open subsets ϑ containing *E*. **[3]**

c) Show that cantor set C has exterior measure zero. **[2]**

Q2) a) Prove that closed sets are measurable. **[5]**

b) If *f* is measurable function and $f = g$ a.e. then prove that *g* is measurable. **[3]**

c) Prove that countable intersection of measurable sets is measurable. **[2]**

Q3) a) State and prove monotone covergence theorem. **[5]**

b) State Littlewood's three principles. **[3]**

c) Define support of a measurable real valued function *f* with suitable example. **[2]**

P.T.O.

Q4) a) Prove that the integral of Lebesgue integral functions is linear, additive, monotonic and satisfy the triangular inequlity. **[5]**

b) Suppose $f(x, y)$ is non-negative mesurable function on $\mathbb{R}^{d^1} \times \mathbb{R}^{d2}$ then prove that for almost every $y \in \mathbb{R}^{d2}$ the slice f^y is measurable on $\mathbb{R}^{d1}$ **[3]**

c) Give an example of non-measurable function. **[2]**

Q5) a) If $E = E_1 \times E_2$ *is* measurable subset of $\mathbb{R}^d$ and $m^*(E_2) > 0$. Then prove that E_1 is measurable , $E_1 \subseteq \mathbb{R}^{d_1}$, $E_2 \subseteq \mathbb{R}^{d_2}$. **[5]**

b) If $\begin{aligned} f(x) &= |x|^{-a} &&, \text{ if } |x| \le 1 \\ &= 0 &&, \text{ otherwise} \end{aligned}$ and $\begin{aligned} g(x) &= |x|^{-b} &&, \text{ if } |x| > 1 \\ &= 0 &&, \text{ otherwise} \end{aligned}$ then show that f is integrable on $\mathbb{R}^d$ if a < d and g is integrable on $\mathbb{R}^d$ if b > d. **[3]**

c) Let $f(t) = \sin t$, $t \in \left[0, \frac{\pi}{2}\right]$. Find total variation of f. **[2]**

Q6) a) Suppose $f(x)$ is integrable on $\mathbb{R}^d$ then prove that f^* satisfies $m\left(\left\{x \in \mathbb{R}^d / f^*(x) > \alpha\right\}\right) = \frac{A}{\alpha}\|f\|_{L^1}\left(\mathbb{R}^d\right)$. **[5]**

b) Suppose F is real valued and bounded variation on $[a, b]$. Then prove that $T_F(a,x) = P_F(a,x) + N_F(a,x)$, $a \le x \le b$. **[3]**

c) If $\int_E f = 0$ and $f(x) \ge 0$ on E then show that $f = 0$ *a.e.* on E. **[2]**

Q7) a) If B = $\{B_1, B_2, \ldots\ldots\ldots, B_N\}$ be a finite collection of open balls in $\mathbb{R}^d$ then prove that there exists a disjoint sub collection $B_{i_1}, B_{i_2}, B_{i_3} \ldots\ldots\ldots, B_{i_k}$ of B such that $M\left(\bigcup_{n=1}^{N} B_i\right) \le 3^d \sum_{j=1}^{k} m\left(B_{ij}\right)$. **[5]**

b) State and prove Egorov's theorem. **[5]**

Q8) a) Define Good Kernels. Suppose ϕ is non negative bounded function in $\mathbb{R}^d$ that is supported on the unit ball $|x| \le 1$ such that $\int_{\mathbb{R}^d} \phi(x)dx = 1$ and define $K_\delta(x) = \frac{1}{\delta}\phi\left(\frac{x}{\delta}\right)$. Show that K_δ is good kernel. **[5]**

b) State and prove rising sun Lemma. **[5]**

Total No. of Questions :8]

SEAT No. :

P2711

[Total No. of Pages :3

[5528]-102
M.A./ M.Sc.
MATHEMATICS
MT-502: Advanced Calculus.
(2013 Pattern) (Semester - I) (Credit System)

Time : 3Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Define directional derivative of a scalar field. Show that the existence of all directional derivatives at a point need not imply continuity of a function at that point. **[5]**

b) Give an example of a function $f(x, y)$ of two variables, which is continuous in each variable separately, but is discontinuous as a function of two variables together. Justify. **[3]**

c) State the Implicit function theorem. **[2]**

Q2) a) State and prove the chain rule for derivatives of vector fields. **[5]**

b) Let $f:\mathbb{R}^2 \to \mathbb{R}^2$ and $g:\mathbb{R}^3 \to \mathbb{R}^2$ be two vector fields defined as $\overline{f}(x, y) = e^{x+2y}\,\overline{i} + \sin(y+2x)\,\overline{j}$;

$\overline{g}(u,v,w) = \left(u + 2V^2 + 3W^3\right)\overline{i} + \left(2V - u^2\right)\overline{j}$ compute the Jacobian matrix $D\overline{h}(1,-1,1)$, where $\overline{h} = \overline{f}.\overline{g}$. **[3]**

c) Find the directional derivative of scalar field $f(x, y, z) = 3x-5y+2z$ at $(2,2,1)$ in the direction of outward normal to the sphere $x^2+y^2+z^2=9$. **[2]**

P.T.O.

Q3) a) State and prove second fundamental theorem of calculus for line integrals. **[5]**

b) Let $\bar{f}$ be a vector field continuous on an open connected set S in $\mathbb{R}^n$. If the line integral of $\bar{f}$ is zero around every piecewise smooth closed path in S, then prove that the line integral of $\bar{f}$ is independent of the path in S. **[3]**

c) Let the two equations $e^u \cos V = x$ and $e^u \sin V = y$ define u and v as functions of x and y, say u = U(x,y) and v = V(x,y). Find explicit formulas for U(x,y) and V(x,y), valid for $x > 0$. **[2]**

Q4) a) Let $\bar{\alpha}$ and $\bar{\beta}$ be equivalent piecewise smooth paths. Prove that, $\int_c \bar{f}.d\bar{\alpha} = \int_c \bar{f}.d\bar{\beta}$ if $\bar{\alpha}$ and $\bar{\beta}$ trace out C in the same direction, and $\int_c \bar{f}.d\bar{\alpha} = -\int_c \bar{f}.d\bar{\beta}$ if $\bar{\alpha}$ and $\bar{\beta}$ trace out c in opposite directions. **[5]**

b) Compute the value of the line integral $\int_c \frac{dx+dy}{|x|+|y|}$, where c is the square with vertices (1, 0), (0, 1), (–1, 0) and (0, –1) traversed once in a counterclockwise direction. **[3]**

c) A force field $\bar{f}$ in 3-space is given by $\bar{f}\,(x, y, z) = x\bar{i} + y\bar{j} + (xz - y)\bar{k}$. Compute the work done by this force in moving a particle from (0, 0, 0) to (1, 2, 4) along the line segment joining these two points. **[2]**

Q5) a) Define step function over rectangle Q. Define double integral of a step function over rectangle Q. If f is a constant function on interior of rectangle Q, Find the double integral of f over Q. **[4]**

b) Evaluate the double integral $\iint_Q y^{-3} e^{tx/y}\, dxdy$, where Q = [0, t]×[1, t], $t > 0$, by repeated integration, given that each integral exists. **[3]**

c) Transform the integral to polar coordinates and compute its value

$$\int_0^{2a} \left[\int_0^{\sqrt{2ax-x^2}} (x^2 + y^2)\,dy \right] dx$$

[The letter a denotes a positive constant] **[3]**

Q6) a) Define a simple parametric surface. If $T = [0, 2\pi] \times \left[0, \frac{\pi}{2}\right]$ under the map $\bar{r}(u,v) = a\cos u \cos v\,\bar{i} + a\sin u \cos v\,\bar{j} + a\sin v\,\bar{k}$ maps to a surface S, find singular points of this surface. Also, explain, whether S is simple. **[5]**

b) Compute the fundamental vector product of

$\bar{r}(u,v) = au\cos v\,\bar{i} + bu\sin v\,\bar{j} + u^2\,\bar{k}$ **[3]**

c) Using surface integral, compute the surface are a of a hemisphere of radius a and centre at origin. **[2]**

Q7) a) Let $\bar{f}(x,y) = P(x,y)\bar{i} + Q(x,y)\bar{j}$ be a vector field that is continuously differentiable on an open simply connected set S in the plane. Prove that $\bar{f}$ is a gradient on S if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ everywhere on S. **[5]**

b) Use a suitable linear transformation to evaluate the double integral $\iint_S e^{(y-x)/(y+x)}.dxdy$, where S is the triangle bounded by the line $x+y=2$ and the two coordinate axes. **[5]**

Q8) a) State and prove Gauss divergence theorem. **[5]**

b) Determine the Jacobian matrix and compute the curl and divergence of $\bar{F}$, where $\bar{F}(x,y,z) = (x^2 + yz)\bar{i} + (y^2 + xz)\bar{j} + (z^2 + xy)\bar{k}$. **[5]**

Total No. of Questions : 8]

SEAT No. :

P2712

[Total No. of Pages : 2

[5528]-103

M.A./M.Sc.

MATHEMATICS

MT-503 : Group Theory

(2013 Pattern) (Semester-I) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) State and prove two step subgroup test. **[5]**

b) For any a and b from a group and any integer n, prove that $\left(a^{-1}ba\right)^{n} = a^{-1}b^{n}a$. **[3]**

c) State and prove Sock-Shoes property. **[2]**

Q2) a) Let $|a| = n$. Then prove that **[4]**

i) $\langle a^{i}\rangle = \langle a^{j}\rangle$ if and only if $\gcd(n,i) = \gcd(n,j)$.

ii) $|a^{i}| = |a^{j}|$ if and only if $\gcd(n,i) = \gcd(n,j)$.

b) Prove that every permutation of a finite set can be written as a cycle or as a product of disjoint cycles. **[4]**

c) Show that A_B contains an element of order 15. **[2]**

Q3) a) For every positive integer n, prove that $\mathrm{Aut}(Zn)$ is isomorphic to U(n). **[5]**

b) Let $\phi : G \to \overline{G}$ be an isomorphism. Then prove that $|a| = |\phi(a)|, \forall a \in G$. **[3]**

c) Find Inn(D_4). **[2]**

P.T.O.

Q4) a) State and prove orbit stabilizer theorem. **[5]**

b) For any group G, prove that $\frac{G}{Z(G)}$ is isomorphic to Inn(G). **[5]**

Q5) a) Determine groups of order 99. **[5]**

b) State and prove G/Z theorem. **[3]**

c) Determine all the homomorphism from Zn to Zn. **[2]**

Q6) a) State and prove first isomorphism theorem. **[5]**

b) Let $\phi : G \to \overline{G}$ be a homomorphism and $g \in G$. Then prove that if $|g|$ is finite then $|\phi(g)|$ divides $|g|$. **[3]**

c) Find all elements of order 5 in $Z_{25} \oplus Z_5$. **[2]**

Q7) a) Let G={1,8,12,14,18,20,27,31,34,38,44,47,51,53,57,64} be the group under multiplication modulo 65. Show that $G \approx Z_4 \oplus Z_4$. **[5]**

b) Define Sylow p-subgroup and state Sylow's three theorems. **[3]**

c) Find conjugacy class of all elements in D_4. **[2]**

Q8) a) Prove that, every group of order p^2, where p is a prime, is isomorphic to Z_{p^2} or $Z_p \oplus Z_p$. **[5]**

b) State Lagrange's theorem. Is the converse of this theorem true? Give justification. **[5]**

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Total No. of Questions : 8]

SEAT No. :

P2713

[Total No. of Pages : 4

[5528]-104
M.A./M.Sc.
MATHEMATICS
MT-504 : Numerical Analysis
(2013 Pattern) (Semester-I) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Solve any five questions out of eight questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Prove that the order of convergence of secant method is approximately 1.618 (α = 1.618) and asymptotic error constant $\lambda \approx C^{1/\alpha} = \left(\dfrac{f''(p)}{2f'(p)}\right)^{(\alpha-1)}$. **[5]**

b) Verify the equation $x^3 - 2x - 5 = 0$ has a root on the real line, perform five iterations of secant method, using $p_0 = 1,\ p_1 = 2$. **[3]**

c) Show that when Newton's method is applied to the equation $\dfrac{1}{x} - a = 0$, the resulting iteration function is $g(x) = x(2 - ax)$. **[2]**

Q2) a) Apply Steffensen method to the iteration function $g(x) = e^{-x}$ using the starting value of $\hat{p}_0 = 0$. Perform four iteration. Compute the absolute error of each approximation and verify that the convergence is quadratic to ten digit, the fixed point is $p = 0.5671432904$. **[5]**

b) Compute each of the following limits and determine the corresponding rate of convergence. **[3]**

i) $\lim\limits_{n\to\infty} \dfrac{n-1}{n^3+2}$

ii) $\lim\limits_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$

c) Show that the order of convergence of Newton's Method is two. **[2]**

P.T.O.

Q3) a) Solve the following system of equation using Gaussian elimination with partial pivoting. **[5]**

$$\begin{aligned} 0.25x + 0.35y + 0.15z &= 0.60 \\ 0.20x + 0.20y + 0.25z &= 0.90 \\ 0.15x + 0.20y + 0.25z &= 0.70 \end{aligned}$$

b) Construct Householder matrix H for $w = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^T$. **[3]**

c) Compute the condition number K_x for the matrix **[2]**

$$A = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

Q4) a) Solve the following system of linear equations by Gauss-Seidel method, start with $x^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ (Perform 3 iterations) **[5]**

$$\begin{aligned} 4x - y &= 2 \\ -x + 4y - z &= 4 \\ -y + 4z &= 10 \end{aligned}$$

b) Solve the following system of linear equations by SOR, start with $x^{(0)} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$, $w = 0.9$ (Perform 2 iterations). **[3]**

$$\begin{aligned} 5x_1 + x_2 + 2x_3 &= 10 \\ -3x_1 + 9x_2 + 4x_3 &= -14 \\ x_1 + 2x_2 - 7x_3 &= -33 \end{aligned}$$

c) Define the terms: **[2]**

i) Rate of covergence

ii) The Lagrange Polynomial $L_{n,j}(x)$

Q5) a) For the matrix $A = \begin{bmatrix} 15 & 7 & -7 \\ -1 & 1 & 1 \\ 13 & 7 & -5 \end{bmatrix}$ with initial vector $x^{(0)} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$.

Perform three iteration of power method to find dominant eigenvalue and corresponding eigenvector. **[5]**

b) Derive the following forward difference approximation for the second derivative. $f''(x_0) \approx \dfrac{f(x_0) - 2f(x_0 + h) + f(x_0 + 2h)}{h^2}$. **[3]**

c) Find the vector valued function F associated with the following system and compute the Jacobian F. **[2]**

$$5\cos x + 6\cos y - 10 = 0$$
$$5\sin x + 6\sin y - 4 = 0$$

Q6) a) Derive the Open Newton-Cotes formula with $n = 2$; **[5]**

$$\int_a^b f(x) = \frac{b-a}{3}\left[2f(a+\Delta x) - f(a+2\Delta x) + 2f(a+3\Delta x)\right],$$

where $\Delta x = \dfrac{b-a}{4}$

b) For a matrix $A = \begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix}$ determine matrices L, U and P such that $LU = PA$ using Gaussian elimination with Scaled partial pivoting. **[3]**

c) If $f(x) = ln(x)$ find $f'(2)$ for $h = 1.0, 0.01$. **[2]**

Q7) a) Verify that the composite midpoint rule has rate of convergence $O(h^2)$ by approximating the value of $\int_0^1 \sqrt{1+x^3}\,dx$. **[5]**

b) Use Householder's method to reduce the following symmetric matrix to tridiagonal form. **[5]**

$$A = \begin{bmatrix} -1 & -2 & 1 & 2 \\ -2 & 3 & 0 & -2 \\ 1 & 0 & 2 & 1 \\ 2 & -2 & 1 & 4 \end{bmatrix}$$

Q8) a) Apply Euler's method to approximate solution of initial value problem, $\frac{dx}{dt} = tx^3 - x,\ 0 \le t \le 1,\ x(0) = 1$, using 4 steps. Find the corresponding error in each step. **[5]**

b) Find solution of the initial value problem, $\frac{dx}{dt} = 1 + \frac{x}{t},\ 1 \le t \le 6,\ x(1) = 1$ using Second order Runge Kutta method with a step size $h = 1$. **[5]**

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Total No. of Questions :8]

SEAT No. :

P2714

[Total No. of Pages : 3

[5528]-105

M.A./M.Sc.

MATHEMATICS

MT 505 : Ordinary Differential Equations

(2013 Pattern) (Semester -I) (Credit system)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions .*
2) *Figures to the right indicate full marks.*

Q1) a) State and prove sturm comparision theorem. **[5]**

b) Test the equation sin ydx - $cos\ xdy = 0$ for exactness and solve it if it exact. **[2]**

c) If K is a given non-zero constant, show that the function $y = ce^{kx}$ is the only solution of the differential equation $\frac{dy}{dx}=ky$ **[3]**

Q2) a) Let $u(x)$ be any non-trivial solution of $u''+q(x)u=0$ where $q(x)>0$ for all $x>0$ and if $\int_1^{\infty} q(x)\,dx=\infty$, then prove that u ($x$) has infinitely many zeros on the positive x-Axis. **[5]**

b) Show that $x=0$ is regular singular point of $2x^2y''+x(2x+1)y'-y=0$ and hence find two independent Frobenius series solution of if. **[5]**

Q3) a) Solve the Euler's equidimensional equation $x^2\,y''+pxy'+qy=0$ by change of independent variable $x=e^z$ where p, q are constants **[5]**

b) Find a particular solution of $y''+y=\sin x$ **[3]**

c) Show that e^x and e^{-x} are linearly independent solutions of $y''-y=0$ on any interval **[2]**

P.T.O.

Q4) a) State and prove sturm separation theorem. **[5]**

b) Find the particular solution of $y'' + y = \text{cosecx}$ by using method of variation of parameters. **[3]**

c) Write general form of legendre's equation and Bessel's equation **[2]**

Q5) a) Show that the function $f(x,y)=xy^2$ satisfies Lips chitz condition on any rectangle $a \le x \le b$ and $c \le y \le d$; but it does not satisfy lips chitz condition on any strip $a \le x \le b$ and $-\infty < y < \infty$ **[5]**

b) Find the general solution near *x=0* of the hypergeometric equation $x(1-x)y'' + [c-(1+a+b)x]y' - aby = 0$ where a,b,c are constants. **[5]**

Q6) a) Show that $e^x = \lim_{b\to\infty} F\left(a, b, \text{a}, \frac{x}{b}\right)$ **[5]**

b) Show that the series $y=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+--$ is a solution of differential equation $y'' + y = 0$ **[3]**

c) Define : Principle of superposition **[2]**

Q7) a) Solve $(1+x)y' = Py$ with $y(0)=1$ by using series method **[5]**

b) Show that the substitution $z = ax + by + c$ changes $y' = f(ax+by+c)$ into an equation with separable variables hence solve $y' = (x+y)^2$ **[3]**

c) Solve the differential equation $(x+y)dx - (x-y)dy = 0$ **[2]**

Q8) a) Solve the following system.

$$\frac{dx}{dt} = x + y$$

$$\frac{dy}{dt} = 4x - 2y$$ **[5]**

b) If $y_1(x)$ and $y_2(x)$ are two solutions of equation $y'' + p(x)y' + Q(x)y = 0$ on an interval [a,b] having common zero in this interval. Then show that $y_1(x)$ and $y_2(x)$ are constant multiples of each other on [a,b] **[3]**

c) State Picard's existance and uniqueness theorem. **[2]**

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Total No. of Questions : 8]

SEAT No. :

P2715

[Total No. of Pages : 3

[5528]-201
M.A./M.Sc.
MATHEMATICS
MT-601 : Complex Analysis
(2013 Pattern) (Semester - II) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Suppose $f = u + iv$ is complex valued function defined on an open set Ω. If u and v are continuously differentiable and satisfy the Cauchy Riemann equations on Ω, then show that f is holomorphic and $f'(z) = \frac{\partial f}{\partial z}$. **[5]**

b) If f is a continuous function on an open set Ω and γ is a curve in Ω, then show that $\left|\int_\gamma f(z)\,dz\right| \leq \operatorname{Sup}_{z\in\gamma}|f(z)|$. length ($\gamma$). **[3]**

c) Show that $4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \Delta$, where Δ is the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. **[2]**

Q2) a) Show that a holomorphic function in an open disc has a primitive in that disc. **[5]**

b) If f is a holomorphic function in a region Ω and $f' = 0$, then show that f is a constant. **[2]**

c) Show that the power series $\sum_{n=1}^{\infty}\frac{z^n}{n}$ converges conditionally at every point of unit circle except at $z = 1$. **[3]**

P.T.O.

Q3) a) If f is a holomorphic function in an open set Ω, then show that f has infinitely many complex derivatives in Ω. Further, show that if $C \subseteq \Omega$ is a circle whose interior is also contained in Ω, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta .$$ **[5]**

b) If f^+ and f^- are holomorphic functions in Ω^+ and Ω^- respectively, that extends continuously to an interval $I \subseteq \mathbb{R}$ and $f^+(x) = f^-(x)$ for all $x \in I$, then show that the function f defined by

$$f(z) = \begin{cases} f^+(z) & \text{if } z \in \Omega^+ \\ f^+(z) = f^-(z) & \text{if } z \in I \\ f^-(z) & \text{if } z \in \Omega^- \end{cases}$$ is holomorphic on all of Ω. **[3]**

c) State Runge's approximation theorem. **[2]**

Q4) a) If Ω is an open set in $\mathbb{C}$ and $T \subseteq \Omega$ is a triangle whose interior is also contained in Ω, then show that $\int_T f(z)dz = 0$, wherever f is holomorphic in Ω. **[5]**

b) Let $F(z, s)$ be defined for $(z, s) \in \Omega \times [0,1]$, where Ω is an open set in $\mathbb{C}$. Suppose F satisfies the following properties : **[5]**

i) $F(z, s)$ is holomorphic in z for each s.

ii) F is continuous on $\Omega \times [0,1]$, then show that f defined on Ω by

$$f(z) = \int_0^1 F(z, s)ds$$ is holomorphic.

Q5) a) Suppose f is holomorphic in a region Ω and $f(z_0) = O$, $z_0 \in \Omega$ and f does not vanish identically in Ω, then show that there exists a neighbourhood U of z_0, a non-vanishing holomorphic function g on U, a unique positive integer n such that $f(z) = (z - z_0)^n g(z)$, for all z in U. **[5]**

b) Show that $\displaystyle\int_0^\infty \frac{1 - \cos x}{x^2} dx = \frac{\pi}{2}$. **[5]**

Q6) a) Suppose that f is holomorphic in an open set Ω containing a circle C and its interior except for a pole z_0 inside C, then show that $\int_C f(z)\,dz = 2\pi i \operatorname{Res}_{z=z_0} f(z)$. **[5]**

b) Show that f has an isolated singularity at a point z_0, then show that z_0 is a pole of f if and only if $|f(z)| \to \infty$ as $z \to z_0$. **[3]**

c) Find the residue of $f(z) = \frac{\tan z}{z^2}$ at $z = \frac{\pi}{2} i$. **[2]**

Q7) a) Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If $|f(z)| > |g(z)|$ for all $z \in$ C, then show that f and $f + g$ have same number of zeros inside the circle C. **[5]**

b) Suppose f is holomorphic in $D_r(z_0) - \{z_0\}$ and has essential singularity at z_0, then show that image of $D_r(z_0) - \{z_0\}$ under f is dense in $\mathbb{C}$. **[3]**

c) Evaluate $\int_{|z|=2} \frac{z^4}{z^5 - 1} dz$. **[2]**

Q8) a) If f is holomorphic and non-constant in a region Ω, then show that f is an open map. **[5]**

b) Find the number of zeros of $f(z) = z^7 - 2z^5 + 6z^3 - z + 1$ in open unit disc. **[3]**

c) State Maximum Modulus theorem. **[2]**

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Total No. of Questions : 8]

SEAT No. :

P2717

[Total No. of Pages : 2

[5528]-203

M.A./M.Sc.

MATHEMATICS

MT-603 : Rings and Modules

(2013 Pattern) (Semester-II) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any Five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) If R is a commutative ring with unity 1 then prove that $A \in Mn(R)$ is a unit if and only if it's determinant, det(A), is a unit in R. **[5]**

b) Show that the element $\overline{5} + \overline{6}x + 1\overline{2}x^2$ is unit in the ring $Z_{24}[x]$. **[3]**

c) Suppose R denotes the matrix ring $M_2(\mathbb{Z})$. Find a matrix $A \in R$ such that A is a zero divisor but not a nilpotent. **[2]**

Q2) a) If $I \subseteq J$ are both two sided ideals in a ring R, then prove that $\left(R/I\right)\Big/\left(J/I\right)$ is naturally isomorphic to R/J. **[4]**

b) Prove that the characteristic of a local ring is either zero or power of a prime. **[4]**

c) Prove or disprove: The ring $Z_6[x]$ is an integral domain. **[2]**

Q3) a) If R is a commutative ring with unity and if $P(x) = a_0 + a_1x + ... + a_rx^r \in R[x]$ is unit in R[x] then prove that a_0 is unit in R and a_1, a_2, ..., a_r are all nilpotent elements in R. **[5]**

b) Let X be a non-empty set. Let P(X) denote the ring of power set of X under addition is the symmetric difference of sets, and multiplication is the intersection of sets. Find units and idempotents of P(X). **[3]**

c) Show that $\dfrac{Q[x]}{\langle x+2 \rangle}$ is a field. **[2]**

P.T.O.

Q4) a) Let a and b are nilpotent elements of a commutative ring R then prove that $a + b$ is nilpotent in R. **[4]**

b) Let $f : \mathrm{R} \to \mathrm{S}$ be a homomorphism of ring. If R and S are commutative then prove that inverse image of a prime ideal in S is a prime ideal in R. **[4]**

c) Show that field of fraction of z[i] is Q[i]. **[2]**

Q5) a) Prove that every Euclidean domain is principal ideal domain. **[5]**

b) Show that $1 + x + x^2 + ... + x^{\mathrm{P}-1}$ is irreducible in $\mathbb{Z}[x]$ for any prime P. **[5]**

Q6) a) Let R be a factorization domain in which every irreducible element is prime. Prove that R is UFD. **[5]**

b) With usual notations prove that $\sqrt{(9)} = \sqrt{(27)} = \sqrt{(3)}$. **[3]**

c) Define local ring. Give an example of non-local ring. **[2]**

Q7) a) For a commutative integral domain R with unity. Prove that the following are equivalent. **[5]**

i) R is field

ii) R[x] is Euclidean domain

iii) R[x] is PID.

b) With usual notation show that $\dfrac{\mathrm{Q}[x]}{\langle 1+x^2 \rangle} \cong \mathrm{Q}[i]$. **[5]**

Q8) a) Show that vector space is a free module. **[5]**

b) State and prove schur's lemma for simple modules. **[5]**

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Total No. of Questions : 8]

SEAT No. :

P2718

[Total No. of Pages : 3

[5528]-204

M.A./M.Sc.

MATHEMATICS

MT-604 : Linear Algebra

(2013 Pattern) (Semester-II) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Solve any five questions out of Eight questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Let V be the vector space of $n \times n$ matrices over F, where char $F \neq 2$. Let V_1 and V_2 be the subsets of symmetric and skew symmetric matrices respectively. Show that V is the direct sum of subspaces V_1 and V_2. **[5]**

b) Let W_1, W_2, W_3 be subspaces of a vector space V such that $W_1 \supset W_2$. Show that $W_1 \cap (W_2 + W_3) = W_2 + (W_1 \cap W_3)$. **[3]**

c) Find a basis of a vector space $\mathbb{C}^2$ over $\mathbb{R}$. **[2]**

Q2) a) If a subset S of V generates V, then there is a subset B of S which is linearly independent and generates V. **[5]**

b) Let V be a finitely generated vector space and let S be a linearly independent subset of V. Then prove that there is a basis of V which contains S. **[3]**

c) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping. Find $f(a, b)$ if $f(1, 0) = (2, 3)$; $f(0, 1) = (-1, 1)$. **[2]**

Q3) a) Let W be a subspace of a finite-dimensional vector space V. Let $\{e_1, e_2, ..., e_m\}$ be a basis of W and let $\{e_1, e_2, ..., e_m, f_1, f_2, ..., f_n\}$ be a basis of V containing that of W. Show that $\{f_1 + W, f_2+W,...., f_n + W\}$ is a basis of V/W. **[5]**

b) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f\left(x_1, x_2, x_3\right) = \left(x_1, x_1 + x_2, x_1 + x_2 + x_3\right)$. Find Im f and ker f. **[3]**

c) Regarding $\mathbb{C}$ and $\mathbb{R}$ as vector spaces over $\mathbb{R}$, write down a basis of $\mathbb{C}/\mathbb{R}$. **[2]**

Q4) a) Let U, V be a vector space over F. Then prove that Hom(U, V) is a vector space over F. Moreover, if dim U = m and dim V = n, then dim Hom $(\mathrm{U}, \mathrm{V}) = m \cdot n$. **[5]**

b) Let $f \in \mathrm{Hom}(\mathrm{V}, \mathrm{V})$. Prove that Im$f$ = Im f^2 if and only if ker$f$ = ker f^2. **[3]**

c) Let $f \in \mathrm{Hom}(\mathrm{V}, \mathrm{V})$. Define a mapping $f^* : \mathrm{V}^* \to \mathrm{V}^*$ by $\left(f^*(\phi)\right)(v) = \phi\left(f(v)\right), v \in \mathrm{V}$. Show that $f^* \in \mathrm{Hom}\left(\mathrm{V}^*, \mathrm{V}^*\right)$. **[2]**

Q5) a) For any given matrix A, prove that there exist inversible matrices P, Q such that $\mathrm{PAQ} = \begin{bmatrix} \mathrm{I}_r & 0 \\ 0 & 0 \end{bmatrix}$. **[5]**

b) Let $\phi \in \mathrm{Hom}\left(\mathrm{V}, \mathrm{V}\right)$ and let $f(t)$ be a polynomial over F such that $f(\phi)=0$. If $f(t)=g(t)\, h(t)$ is a factorization of $f(t)$ into relatively prime polynomials $g(t)$, $h(t)$, then prove that $\mathrm{V} = \ker g(\phi) \oplus \ker h(\phi)$. **[3]**

c) Show that if A is non zero idempotent then 0, 1 are the only eigenvalues of A. **[2]**

Q6) a) Reduce the following matrix to Jordan canonical form: **[5]**

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}.$$

b) If λ, μ are distinct eigenvalues of A, then prove that $\ker(A - \lambda I)^p \subseteq \operatorname{Im}(A - \mu I)^q$ for all positive integers p and q. **[3]**

c) Define left radical and right radical of a bilinear form B. **[2]**

Q7) a) Let H be a hermitian form on V. Then a linear mapping T of V into itself is H-unitary if and only if $H(Tx, Tx) = H(x, x), \forall\, x \in V$. **[5]**

b) State and prove principal axis theorem. **[5]**

Q8) a) Reduce the quadratic form $4x^2 + y^2 - 8z^2 + 4xy - 4xz + 8yz$ to the diagonal form $\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2$ by an orthogonal transformation of co-ordinates. **[5]**

b) Reduce the equation $x^2 + y^2 + z^2 - 4xy - 4yz - 4zx - 3 = 0$ to its standard form by rotation and translation of axes, and identify the quadratic surface. **[5]**

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Total No. of Questions : 8]

SEAT No. :

P2719

[Total No. of Pages : 2

[5528] - 205
M. A. / M. Sc.
MATHEMATICS
MT- 605: Partial Differential Equations
(2013 Pattern) (Semester - II) (Credit System)

Time : 3 Hours] *[Max. Marks :50*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Eliminate the parameters a and b from the equation
$z^2 (1 + a^3) = 8(x + ay + b)^3$ **[4]**

b) Find the general solution of $(z^2 - 2yz - y^2)\, p + x\,(y + z)\, q = x\,(y - z)$. **[4]**

c) State the conditions for the equations f(x,y,z,p,q) = 0 and $g(x,y,z,p,q) = 0$ to be compatiable on the domain D. **[2]**

Q2) a) Find the general solution of
$(y^2 + yz + z^2)\, dx + (z^2 + xz + x^2)\, dy + (y^2 + xy + x^2)\, dz = 0$. **[4]**

b) Show that the equations $xp = yq$ and $z\,(xp + yq) = 2xy$ are compatible. Also find their common solution. **[4]**

c) Find the complete integral of $p^2 + q^2 = x + y$. **[2]**

Q3) a) If $h_1 = 0$ and $h_2 = 0$ are compatible with f = 0, then prove that h_1 and h_2 satisfy:

$$\frac{\partial(f,h)}{\partial(x,u_x)} + \frac{\partial(f,h)}{\partial(y,u_y)} + \frac{\partial(f,h)}{\partial(z,u_z)} = 0$$ **[4]**

where $h_{i,i = 1,2}$

b) Find the complete integral of the equation $z^2 = pqxy$ by Charpit's method. **[4]**

c) Verify that the equation is integrable $yz\,(y+z)dx + xz(x+z)dy + xy\,(x+y)dz = 0$. **[2]**

P.T.O.

Q4) a) Find the general integral of the differential equation $(x-y)y^2p+(y-x)x^2q = (x^2+y^2)z$ and the particular solution through curve C $xz = a^2, y = 0$ **[4]**

b) Find by the method of characteristic, the integral surface of $pq = z$ which passes through curve $x_0 = 0, y_0 = s, z_0 = s^2$ **[4]**

c) Derive the analytic expression for the Monge cone at (x_0, y_0, z_0). **[2]**

Q5) a) Verify that the equation is integrable and find the corresponding integral $\left(y^2 + yz\right)dx + \left(z^2 + xz\right)dy + \left(y^2 - xy\right)dz = 0$. **[4]**

b) Find d' Alembert's solution of one dimensional wave equation which describes the vibration of infinite string. **[3]**

c) Reduce the equation

$U_{xx} + 2U_{xy} + 17U_{yy=0}$ to canonical form and solve it. **[3]**

Q6) a) State and prove Kelvin's inversion theorem. **[5]**

b) Find the solution of the Heat – equation in an infinite rod which is defined as: $U_t = kU_{xx} \quad , \quad -\infty < x < \infty, t > 0$

$$U(x,0) = f(x) \quad , \quad -\infty < x < \infty$$ **[5]**

Q7) a) If U (x,y) is harmonic in a bounded domain D and continuous in $\bar{D} = D \bigcup B$. Then U attains It's maximum on the boundary B of D. **[4]**

b) State and prove Harnack's theorem. **[4]**

c) Classify the following equation into hyperbolic, parabolic or elliptic type $u_{xx} + 2(1+\alpha y)u_{yz} = 0$. **[2]**

Q8) a) Use Duhamel's principle and solve the non homogeneous wave

equation $\quad u_{tt} - c^2u_{xx} = F(x,t), -\infty < x < \infty, t > 0$

with conditions $\quad u(x,0) = u_{t(x,0)} = 0 - \infty < x < \infty$. **[5]**

b) State Dirichlet's problem for rectangle and find it's solutions. **[5]**

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Total No. of Questions :8]

SEAT No. :

P2720

[Total No. of Pages :3

[5528]-301
M.A./ M.Sc.
MATHEMATICS
MT-701: Combinatorics
(2013 Pattern) (Credit System) (Semester - III)

Time : 3Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) A computer password consists of 3 alphabets followed by 3 or 4 digits, find

i) The total number of passwords that can be created,

ii) The number of passwords in which no alphabet and no digit repeats. **[5]**

b) Given 5 distinct pair of gloves, 10 distinct gloves in all, how many ways are there to distribute two gloves to each of 5 girls, if each girl gets one left had glove and one right hand glove? **[3]**

c) How many five digit numbers are there that are the same when the order of their digits is inverted? **[2]**

Q2) a) How many arrangements of letters in REPETITION are there with first E occurring before the first T? **[5]**

b) Prove by a combinatorial argument **[3]**

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\_\_\_\_\_\_\_\_\_\_+\binom{n}{n}=2^n$$

c) A student must answer 5 out of 10 questions on a test,. including at least 2 of the first 5 questions. How many subsets of 5 questions can be answered? **[2]**

P.T.O.

Q3) a) Show with generating function that every positive integer can be written as a unique sum of distinct powers of 2. **[5]**

b) Find the coefficient of x^{12} in $\dfrac{x+3}{1-2x+x^2}$. **[3]**

c) Find the generating function for a_r the number of ways to distribute r identical objects into four different boxes with between three and six objects in each box. **[2]**

Q4) a) How many r-digit ternery sequences are there with **[5]**

i) An even number of zeros?

ii) An even number of zeros and even number of ones?

b) Find a recurrence relation for the amount of money in a saving account after n years, if the interest rate is 6 percent/year, and Rs.50 is added to the start of each year. **[3]**

c) Find the exponential generating function for the number of ways to distribute r people into six different rooms with between two and four in each room. **[2]**

Q5) a) Find ordinary generating function whose coefficient $a_r=3r^2$. Hence evaluate the sum $0+3+12+..........+3n^2$. **[5]**

b) Show that an = $An^{\log 2c} + \left(\dfrac{2d}{2-c}\right)n$ be a solution of recurrence relation

$a_n = c\, a_{n/2} + dn$. where A,c,d are constants with C>2. **[5]**

Q6) a) Solve the recurrence relation $an=2a_{n-1}+(-1)^n$, $a_0=2$. **[4]**

b) How many arrangements of the digits 0,1,2,............9 are there that do not end with an 8 and do not begin with a 3? **[4]**

c) Show that $\binom{n}{1}+6\binom{n}{2}+6\binom{n}{3}=n^3$ **[2]**

Q7) a) Solve the recurrence relation an=a_{n-1}+2 using generating function. **[4]**

b) Use Inclusion-Exclusion formula to find number of m-digit decimal sequences (using digits 0,1,2,............9) in which digits 1,2,3 all appear. **[4]**

c) How many numbers greater than 30,00,000 can be formed by arrangements of 1,1,2,2,4,6,6,? **[2]**

Q8) a) What is the probability that if n people randomely reach into a dark close at to retrieve their hats, no person will pick his own hat? **[5]**

b) How many ways are there to send six different birthday cards, denoted C_1,C_2,C_3,C_4,C_5,C_6 to three aunts and three uncles, denoted by A_1,A_2,A_3, U_1,U_2,U_3 if aunt A_1 would not like Card C_2 and C_4; if A_2 would not like C_1 and C_5; If A_3 likes all cards if u_1 would not like C_1 and C_5; U_2 would not like C_4; and if U_3 would not like C_6. **[5]**

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Total No. of Questions :8]

SEAT No. :

P2721

[Total No. of Pages :2

[5528]-302
M.A./M.Sc.
MATHEMATICS
MT-702: Field Theory
(2013 Pattern) (Semester - III) (Credit System)

Time : 3Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks***

Q1) a) Let F be a field and $p(x) \in F[x]$ be an irreducible polynomial of degree n over F. Let k=F (x)/$\langle p(x)\rangle$ and $\theta = x \bmod(\langle p(x)\rangle) \in k$. Then prove that the set $\{1, \theta, \theta^2 \ldots\ldots\ldots\ldots \theta^{n-1}\}$ is a basis for the vector space K over F. **[5]**

b) Show that a field k generated over F by a finite number of algebraic elements of degree $n_1, n_2, \ldots\ldots\ldots\ldots n_k$. is algebraic of degree less than or equal to $n_1, n_2 \ldots\ldots\ldots\ldots n_k$. **[3]**

c) Find the minimal polynomial of $\alpha = \sqrt{2} - \sqrt{3}$ over Q. **[2]**

Q2) a) Show that any quadratic extension k of any field F of characteristic not equal to 2 is Galois. **[5]**

b) Find the smallest extension of Q having root of polynomial $x^4 - 2$. **[2]**

c) Let F be a field and α,β are algebraic over F then prove that $\alpha \pm \beta, \alpha\beta \, and \, \frac{\alpha}{\beta}(\beta \neq o)$ are all algebraic over F. **[3]**

Q3) a) Show that Galois group of $x^{p^n} - x$ over $\mathbb{F}_p$ is a cyclic group of order n. **[5]**

b) Show that $x^3 - \sqrt{2}$ is irreducible polynomial over $Q(\sqrt{2})$. **[3]**

c) If F is a field and $\overline{F}$ is an algebraic closure of F then prove that $\overline{(\overline{F})} = \overline{F}$. **[2]**

P.T.O.

Q4) a) If the degree of the extension K/F is a prime p then show that any subfield E of k containing F is either K or F. **[5]**

b) If F is a field of characteristic p and $f(x) \in F[x]$ then prove that $D_x(f(x)) = 0$ if and only if $f(x) = g(x^p)$ for some $g(x) \in \mathrm{F}[x]$. **[3]**

c) Is $f(x)=(x^2-2)^n$ separable over Q? Justify. **[2]**

Q5) a) Show that the degree of extension of the splitting field for $x^3 - 2 \in Q[x]$ is 6. **[5]**

b) Find the discriminant of the polynomial $f(x)=x^3-2x+4$. Is the Galois group of $f(x)$ solvable? Justify. **[3]**

c) Define the p^{th} cyclotomic polynomial $\Phi_n(x)$. Find $\phi_n(x)$ for n=1,2,3. **[2]**

Q6) a) Define elementary symmetric function and state the fundamental theorem on symmetric function. **[5]**

b) Show that cos20° and sin20° can not be constructed by using straightedge and compass. **[3]**

c) Find a fixed field of $\mathrm{Aut}\left(Q\left(\sqrt{2}\right)/\mathrm{Q}\right)$. **[2]**

Q7) a) If $G = \{\sigma_1 = 1, \sigma_2, ..., \sigma_n\}$ be a subgroup of automorphisms of a field and F be the fixed field then prove that $[K:F]=|G|$. **[5]**

b) Show that any two splitting fields for a polynomial $f(x) \in F[x]$ over a field F are isomorphic. **[5]**

Q8) a) Find a Galois group of the polynomial $f(x) = x^3 - 2 \in Q[x]$. **[5]**

b) Prove that cyclotomic polynomial $\Phi_n(x)$ is a monic polynomial in $\mathbb{Z}[x]$ of degree Φ (n) **[5]**

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Total No. of Questions : 8]

SEAT No. :

P2722

[Total No. of Pages : 2

[5528]-303
M.A./M.Sc.
MATHEMATICS
MT-703 : Functional Analysis
(2013 Pattern) (Semester-III) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicates full marks.*

Q1) a) If H is a C-Hilbert space and $A \in B(H)$, then prove that A is hermitian if and only if $\langle Ah, h \rangle \in \mathbb{R}$ for all h in H. **[5]**

b) Define unilateral shift, and find its adjoint. **[3]**

c) Define Volterra operator. **[2]**

Q2) a) Prove that the Bergman space for a open set G of $\mathbb{C}$ is a Hilbert space. **[5]**

b) If A is a subset of Hilbert space H, then prove that $\left(A^{\perp}\right)^{\perp}$ is the closed linear span of A in H. **[3]**

c) Give an example of a convex set in a Hilbert space. **[2]**

Q3) a) State and prove Riesz representation theorem for a Hilbert space. **[5]**

b) Two Hilbert spaces are isomorphic if and only if they have the same dimensions. **[5]**

Q4) a) Prove that an operator T on a Hilbert space H is compact if and only if T^* is compact. **[5]**

b) If T is a compact operator on a Hilbert space H, $\lambda \neq 0$ and $\inf\left\{\|(T-\lambda)h\| / \|h\|=1\right\}=0$, then prove that λ is an eigenvalue of T. **[3]**

c) Give an example of idempotent on a Hilbert space which is not an projection. **[2]**

P.T.O.

Q5) a) State and prove the uniform boundedness principle. **[5]**

b) If X is a normed space, then prove that X^* is a Banach space. **[5]**

Q6) a) State and prove closed graph theorem. **[5]**

b) Let X and Y are normed spaces with $\dim X < \infty$. If $T : X \to Y$ is a linear transformation, then prove that T is continuous. **[3]**

c) State Hahn-Banach theorem. **[2]**

Q7) a) If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on the vector space X, then prove that these norms are equivalent if and only if there are positive constants c and C such that $c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$. **[5]**

b) If A is a normal operator and $\lambda \in \mathbb{F}$, then prove that $\ker(A-\lambda) = \ker(A-\lambda)^*$, and $\ker(A-\lambda)$ is reducing subspace for A. **[3]**

c) Define reflexive space and give an example. **[2]**

Q8) a) Let $\phi : [-\pi, \pi] \to \mathbb{C}$ be defined by $\phi(t) = e^{it}$ and $M_\phi : L^2[-\pi, \pi] \to L^2[-\pi, \pi]$ be the operator defined by $M_\phi(f) = \phi f$. Find M_ϕ^* and $\|M_\phi\|$. **[5]**

b) Let $M = \{\{x_n\} \in l^2 / x_n = 0 \text{ for all but finitely many } n\}$. Is M a closed subspace of l^2? Justify. **[3]**

c) State spectral theorem for compact self-adjoint operator. **[2]**

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Total No. of Questions : 8]

SEAT No. :

P2723

[Total No. of Pages : 2

[5528]-401
M.A./M.Sc.
MATHEMATICS
MT-801 : Number Theory
(2013 Pattern) (Semester - IV) (Credit System)

Time : 3 Hours] ***[Max. Marks : 50***

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) If $(a, m) = 1$ then prove that $a^{\phi(m)} \equiv 1 \pmod{m}$. **[5]**

b) Show that the congruence $x^2 \equiv 1 \pmod{2^\alpha}$ has one solution when $\alpha = 1$, two solutions when $\alpha = 2$ and precisely the four solutions 1, $2^{\alpha-1} - 1$, $2^{\alpha-1} + 1$, -1 when $\alpha \geq 3$. **[3]**

c) Is $x^2 \equiv -2 \pmod{61}$ has solution? Justify. **[2]**

Q2) a) If p is an odd prime and $(a, 2p) = 1$ then prove that $\left(\frac{a}{p}\right) = (-1)^t$ where

$$t = \sum_{j=1}^{(p-1)/2} \left[\frac{ja}{p}\right].$$ **[5]**

b) Show that the product of three consecutive integers is divisible by 504 if the middle one is cube. **[3]**

c) Let $f(x) = x^2 + x + 7$. Find all roots of the congruence $f(x) \equiv 0 \pmod{15}$. **[2]**

Q3) a) State and prove Mobius inversion formula. **[5]**

b) What is the highest power of 2 dividing 533!? **[3]**

c) Is $22x + 40y = 1$ has solution? Justify. **[2]**

P.T.O.

Q4) a) State and prove Wilson's theorem. **[5]**

b) Prove that $\sum_{d/n} d = \sum_{d/n} \frac{n}{d}$. **[3]**

c) Find the value of the Legendre symbol $\left(\frac{2}{5}\right)$. **[2]**

Q5) a) Prove that the number $\sum_{j=1}^{\infty} 10^{-j!}$ is transcendental. **[5]**

b) Prove that the quadratic residues modulo p are congruent to $1^2, 2^2, 3^2, ..., \{(p-1)/2\}^2$ where p is an odd prime. **[5]**

Q6) a) Let a, b and c be integers with not both a and b equal to 0 and let g = g.c.d. (a, b). If $g \nmid c$ then show that the equation $ax + by = c$ has no solution in integers. If $g|c$ then show that $ax + by = c$ has infinitely many solutions and are of the form $x = x_1 + \frac{kb}{g}$, $y = y_1 - \frac{ka}{g}$ where k is an integer and (x_1, y_1) is one integral solution. **[5]**

b) Find all integers that give the remainders 1, 0, 5 when divided by 4, 3, 7 respectively. **[5]**

Q7) a) If ξ is an algebraic number of degree n then prove that every number in $\mathbb{Q}(\xi)$ can be written uniquely in the form $a_0 + a_1\xi + ... + a_{n-1}\xi^{n-1}$ where the a_i are rational numbers. **[5]**

b) Prove that the number of positive irreducible fractions ≤ 1 with denominator $\leq n$ is $\phi(1)+\phi(2)+......+\phi(n)$. **[3]**

c) Prove that $1 + i$ is a prime in $\mathbb{Q}(i)$. **[2]**

Q8) a) State and prove Gaussian Reciprocity Law. **[5]**

b) Prove that $\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)=0$, p is an odd prime. **[3]**

c) Find $d(14)$, $\sigma(14)$. **[2]**

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Total No. of Questions : 8]

SEAT No. :

P2725

[Total No. of Pages : 3

[5528]-403

M.A./M.Sc.

MATHEMATICS

MT-803 : Fourier Analysis and Boundary Value Problems

(2013 Pattern) (Semester-IV) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Let f denote a function such that (i) f is continuous on the interval $-\pi \le x \le \pi$ (ii) $f(-\pi) = f(\pi)$ (iii)Its derivative f' is piecewise continuous on the interval $-\pi < x < \pi$. If a_n and b_n are Fourier coefficients $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$ for function f, then prove that Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ converges absolutely and uniformly to $f(x)$ on the interval $-\pi \le x \le \pi$. **[5]**

b) Find the Fourier series for the function $f(x) = x(-\pi < x < \pi)$. **[3]**

c) Find the Fourier cosine series for the function $f(x) = \pi - x(0 < x < \pi)$.**[2]**

Q2) a) If a function G(u) is piecewise continuous on the interval $(0 < u < \pi)$, then prove that $\lim_{N\to\infty}\int_0^{\pi} G(u)\sin\left(\frac{u}{2} + Nu\right) du = 0$ where N denotes positive integers. **[5]**

b) Find the Fourier sine series for the function $f(x) = x(0 < x < 1)$. **[3]**

c) Prove or disprove all Fourier series are differentiable. **[2]**

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Q3) a) Solve the following boundary value problem. **[5]**

$$y_{tt}(x,t) = a^2 y_{xx}(x,t) \; (0 < x < c,\, t > 0)$$
$$y(0,t) = 0,\; y(c,t) = 0,\; y_f(x,0) = 0$$
$$y(x,0) = f(x) \; (0 \le x \le c)$$

b) Solve the following boundary value problem. **[5]**

$$u_t(x,t) = k\, u_{xx}(x,t) \; (0 < x < \pi,\, t > 0)$$
$$u(0,t) = 0,\; u(\pi,t) = 0,\; u(x,0) = f(x)$$

Q4) a) Solve the following boundary value problem. **[5]**

$$u_{xx}(x,y) + u_{yy}(x,y) = 0 \; (0 < x < a,\, 0 < y < b)$$
$$u(0,y) = 0,\; u(a,y) = 0 \; (0 < y < b)$$
$$u(x,0) = f(x),\; u(x,b) = 0 \; (0 < x < a)$$

b) Solve the following boundary value problem. **[5]**

$$\rho^2 u_{\rho\rho}(\rho,\phi) + \rho u_{\rho}(\rho,\phi) + u_{\phi\phi}(\rho,\phi) = 0 \; (1 < \rho < b,\, 0 < \phi < \pi)$$
$$u(\rho,0) = 0,\; u(\rho,\pi) = 0 \; (1 < \rho < b)$$
$$u(1,\phi) = 0,\; u(b,\phi) = u_0 \; (0 < \phi < \pi)$$

Q5) a) Prove that a necessary and sufficient condition for an orthonormal set $\{\phi_n(x)\}(n = 1,2,3....)$ to be complete is that for each function *f* in the space considered Parseval's equation $\sum_{n=1}^{\infty} C_n^2 = \|f\|^2$ where C_n are the Fourier constants $C_n = (f, \phi_n)$ be satisfied. **[5]**

b) Show that the function $\psi_1(x) = 1$ and $\psi_2(x) = x$ are orthogonal on the interval $-1 < x < 1$ and determine constants A and B such that $\psi_3(x) = 1 + Ax + Bx^2$ is orthogonal to both $\psi_1(x)$ and $\psi_2(x)$ on the interval. **[3]**

c) Show that each of the functions $y_1 = \frac{1}{x}$ and $y_2 = \frac{1}{1+x}$ satisfies the nonlinear differential equation $y' + y^2 = 0$. Then show that the sum $y_1 + y_2$ fails to satisfy that equation. **[2]**

Q6) a) If λ_m and λ_n are distinct eigenvalues of the Sturm-Liouville problem $\left[r(x)X'(x)\right]' + \left[q(x)+\lambda p(x)\right]X(x)=0 \quad (a<x<b)$ under the condition $a_1 X(a)+a_2 X'(a)=0$, $b_1 X(b)+b_2 X'(b)=0$, then prove that corresponding eigen functions $X_m(x)$ and $X_n(x)$ are orthogonal with respect to weight function $p(x)$ on the interval $a<x<b$. **[5]**

b) Find the eigenvalues and the normalized eigen functions of Sturm-Liouville problem. $X''+\lambda X=0,\ X(0)=0,\ hX(1)+X'(1)=0\ (h>0)$. **[3]**

c) If m and n are positive integers, then show that **[2]**

$$\int_0^{\pi} \cos mx \cos nx \; dx = \begin{cases} 0 & \text{when } m \neq n \\ \dfrac{\pi}{2} & \text{when } m = n \end{cases}$$

Q7) a) Solve the Bessel's differential equation $x^2 y'' + xy' + \left(x^2 - n^2\right)y = 0$. **[5]**

b) Establish the recurrence relation **[3]**

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x) \quad (n=0,1,2,...).$$

c) Show that $\sqrt{1}=1$ and verify that $\sqrt{(n+1)} = n!$ when $n=0,1,2,3,....$ **[2]**

Q8) a) Prove that the eigen values and corresponding eigen functions of singular Sturm-Liouville problem $\left[\left(1-x^2\right)X'(x)\right]' + \lambda X'(x) = 0 \ (-1<x<1)$ are $\lambda_n = n(n+1)$ and $X_n(x) = P_n(x)\ (n=0,1,2,...)$ where $P_n(x)$ are Legendre polynomials. Also prove that the set $\{P_n(x)\}(n=0,1,2,...)$ is orthogonal on the interval $-1<x<1$, with weight function unity. **[5]**

b) Derive the Rodrigue's formula $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left(x^2-1\right)^n$. **[3]**

c) Verify that the Legendre polynomials $P_2(x) = \frac{1}{2}\left(3x^2-1\right)$, $P_3(x) = \frac{1}{2}\left(5x^3-3x\right)$ form an orthogonal set on the interval $-1<x<1$. **[2]**

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