Total No. of Questions :8]

SEAT No. :

P2049

[Total No. of Pages :2

[4821]-11

M.A. / M.Sc.

MATHEMATICS

MT-501: Real Analysis - I

(2008 Pattern) (Semester - I)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Define an inner product space and show that $\mathbb{R}^2$ is an inner product space with respect to the inner product defined by, $\langle x, y\rangle = \sum_{i=1}^{2} x_i y_i$. **[6]**

b) State and prove Cauchy Schwarz inequality. **[6]**

c) With usual notations find $d(1,1),(2,3)$ in $\|\cdot\|_1$ and $\|\cdot\|_2$. **[4]**

Q2) a) Show that $\|x\| = \frac{1}{3}\|x\|_1 + \frac{2}{3}\|x\|_2$ defines a norm on $\mathbb{R}^n$. **[6]**

b) Prove that l^1 is infinite dimensional space. **[6]**

c) Let $f(x) = 1$, $g(x) = x$ defined on [0, 1] then with respect to sup norm in $e[0, 1]$ find $d(f, g)$. **[4]**

Q3) a) Show that compact sets of a metric space are closed. **[6]**

b) Define a separable metric space and show that reals with discrete metric is not separable. **[6]**

c) State **[4]**

i) Heine Borel theorem

ii) Arzela Ascoli theorem

Q4) a) Let A and B be subsets of a metric space (X, a) prove or disprove **[6]**

i) $\text{int}\,(A \bigcup B) = \text{int}(A) \bigcup \text{int}(B)$

ii) $\text{int}\,(A \bigcap B) = \text{int}(A) \bigcap \text{int}(B)$

b) i) Let $A = (0,1) = \{(x,0)/0 < x < 1\} \subset \mathbb{R}^2$ then whether A is open, closed, neither or both? Justify. **[6]**

ii) Find closure of set of rational numbers in $\mathbb{R}$ with usual metric.

c) Prove that, in any metric space a Cauchy sequence is a bounded sequence. **[4]**

Q5) a) Define a measurable function and show that addition and multiplication of two measurable functions are also measurable. **[6]**

b) Let $A, B \subset \mathbb{R}^2$ defined by $A = [0,4] \times (1,10]$ $B = (0,1] \times [0,2]$. Draw picture of S(A, B) and find D (A, B). **[6]**

c) Give an example to show that strict inequality hold in fatou's lemma. **[4]**

Q6) a) State and prove Lebesque dominated convergence theorem. **[6]**

b) Let $1 < p < \infty, 1 < q < \infty$ and $f \in L^p, g \in L^q$ then show that $fg \in L^1$ and $\|fg\|_1 \leq \|f\|'_p \|g\|_q$. **[6]**

c) Prove that step functions are dense in $L^p(u)$ for $1 \leq p < \infty$. **[4]**

Q7) a) State and prove Bessel's inequality. **[8]**

b) Calculate Fourier series expansion for $f(x) = x^2$. **[8]**

Q8) a) Obtain Legendre - polynomials by applying Gram-Scmidt process to functions 1, x, x^2,... **[8]**

b) Show that the sequence $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} / n \in z \right\}$ is a compute orthonormal sequence in $L^2[-\pi, \pi]$ **[8]**

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Total No. of Questions :8]

SEAT No. :

P2050

[Total No. of Pages :3

[4821]-12

M.A. / M.Sc.

MATHEMATICS

MT-502: Advanced Calculus

(2008 Pattern) (Semester - I) (Old Course)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Let *f* be a scalar field of vector variable **[8]**

i) What is the derivative of *f* with respect to a vector?

ii) What is the directional derivative of *f* at a vector along some direction of a vector?

iii) What is total derivative of *f* at a given vector?

Explain the significance and difference of the above terms.

b) A scalar field *f* is defined on $\mathbb{R}^n$ by the equation $f(\overline{x}) = \overline{a}, \overline{x}$ where $\overline{a}$ is a constant vector. Compute $f'(\overline{x}, \overline{y})$ for arbitrary $\overline{x}$ and $\overline{y}$. **[4]**

c) Let f is a scalar field on $\mathbb{R}^n$ and g is a real valued function of real variable such that $g(t) = f(\overline{a} + t\overline{y})$. If one of the derivative $g'(t)$ or $f'(\overline{a} + t\overline{y}, \overline{y})$ exists then prove that the other also exists and are equal. **[4]**

Q2) a) State and prove chain rule for the derivatives of a vector fields. **[8]**

b) Define the gradient vector of a scalar field f at $\overline{a}$. Find the gradient vector for the scalar field $f(x, y) = x^2 + y^2 \sin(xy)$ at $\left(1, \frac{\pi}{2}\right)$. **[4]**

c) Let z be the function of u and v where $u = x^2 - y^2 - 2xy$ and $v = y$ find $(x + y)\frac{\partial z}{\partial x} + (x - y)\frac{\partial z}{\partial y}$. **[4]**

Q3) a) Define line integral. Show by an example that the line integral may depends on the path joining the two points. **[6]**

b) Compute the massM of one coil of a spring having the shape of the helix whose vector equation is $\bar{\alpha}(t) = a\cos t\,\bar{i} + a\sin t\,\bar{j} + bt\,\bar{k}$ if the density at (x, y, z) is $x^2 + y^2 + z^2$. **[5]**

c) Calculate the line integral of the vector field

$f(x, y) = (x^2 - 2xy)\bar{i} + (y^2 - 2xy)\bar{j}$ from $(-1, 1)$ to $(1, 1)$ along the parabola $y = x^2$. **[5]**

Q4) a) State and prove the first fundamental theorem of calculus for line integrals. **[6]**

b) Let $\bar{f}$ be a vector field continuous on an open connected set S in $\mathbb{R}^n$. If the line integral of $\bar{f}$ is zero around every piecewise smooth close path in S then prove that the line integral of $\bar{f}$ is independent of the path in S. **[6]**

c) Determine the work done by constant force $\bar{f}$ in moving a particle from a point $\bar{a}$ to a point $\bar{b}$ along any piecewise smooth path joining $\bar{a}$ and $\bar{b}$. **[4]**

Q5) a) Let f be bounded on a rectangle Q in $\mathbb{R}^2$. Show that upper integral $\bar{I}(f)$ and lower integral $\underline{I}(f)$ exist. Also prove that f is integrable over Q if and only if $\bar{I}(f) = \underline{I}(f)$. **[6]**

b) State only the general formula for change of variables in double integrals. Explain the notation used. **[5]**

c) Transform the integral to polar co-ordinates and compute its value **[5]**

$$\int_0^1 \left[\int_{x^2}^{x} (x^2 + y^2)^{-1/2} dy \right] dx.$$

Q6) a) State and prove Green's theorem for plane regions bounded by smooth Jordan curves. **[6]**

b) Use Green's theorem to evaluate the line integral $\oint_C y^2 dx + x\, dy$ where C is the square with vertices (0, 0), (2, 0), (2, 2), (0, 2). **[5]**

c) Evaluate the triple integral $\iiint_S xy^2 z^3 dx\, dy\, dz$ where S is the solid bounded by the surface z = xy and the planes $y = x$, $x = 1$ and $z = 0$. **[5]**

Q7) a) Define a simple parametric surface. If $T = [0, 2\pi] \times [0, \pi/2]$ maps under $\bar{r}(u,v) = a\cos u \cos v\, \bar{i} + a\sin u \cos v\, \bar{j} + a\sin v\, \bar{k}$ to a surface S, find singular points of this surface. Also explain whether S is simple. **[6]**

b) Define the surface integral and explain the terms involved in it. **[6]**

c) Let S be a parametric surface whose vector representation is $\bar{r}(u,v) = (u+v)\,\bar{i} + (u-v)\,\bar{j} + (1-2u)\,\bar{k}$. Find the fundamental vector product and the unit normal to the surface. **[4]**

Q8) a) State and prove Gauss divergence theorem. **[8]**

b) Let S be the surface of unit cube $0 \le x \le 1$ $0 \le y \le 1$ $0 \le z \le 1$ and let $\bar{n}$ be the unit outer normal to S. If $\bar{F}(x,y,z) = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$, use the divergence theorem to evaluate the surface integral $\iint_S \bar{F}, \bar{n}\, ds$. **[6]**

c) Determine the Jacobian matrix of the vector field

$$\bar{F}(x,y,z) = (x^2 + yz)\bar{i} + (y^2 + xz)\bar{j} + (z^2 + xy)\bar{k}$$ **[2]**

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Total No. of Questions :8]

SEAT No. :

P2051

[Total No. of Pages :3

[4821]-13

M.A. / M.Sc.

MATHEMATICS

MT-503: Linear Algebra

(2008 Pattern) (Semester - I)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Show that a linearly independent set in a finite dimensional vector space V can be extended to a basis of V. Extend the set {(1, 1, 0), (1, 1, 1)} to a basis of $\mathbb{R}^3$. **[6]**

b) Find dimensions of the following subspaces of the vector space $\mathbb{R}^{n\times n}$ of all $n\times n$ matrices over $\mathbb{R}$: **[5]**

$W_1 = \{A \in \mathbb{R}^{n\times n} \mid A = A^t\}$; $W_2 = \{A \in \mathbb{R}^{n\times n} \mid A = -A^t\}$; $W_3 = \{A \in \mathbb{R}^{n\times n} \mid \text{trace } A = 0\}$.

c) Prove that the set of points on any circle in $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^2$. Describe all subspaces of $\mathbb{R}^2$. **[5]**

Q2) a) Let V and W be vector spaces over K of dimensions n and m, respectively. Prove that the vector space (L(V, W) of all linear transformations from V to W has dimension mn. **[6]**

b) Define a nilpotent operator. Show that if T is a nilpotent operator on V, then T - I is an invertible operator on V. **[5]**

c) Give a one-one linear map $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$. Can this map be onto? Justify. **[5]**

P.T.O.

Q3) a) Let $T:V \to W$ be a linear map. prove that V / ker T is isomorphic to Im T. Identify the space V / ker T, where V= $\mathbb{R}^2$ and $T:\mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x, y) = (x, 0)$. **[6]**

b) Let T be a linear operator on a vector space V such that $T^2 = 2015T$. Prove that V is the direct sum of ker T and *im* T. **[5]**

c) Prove that $B = \{1, 1 + x, 1 + x + x^2\}$ is a basis for the vector space $\mathbb{R}_2[x]$ of polynomials over $\mathbb{R}$ upto degree 2. Find the matrix representation of the differential operator on $\mathbb{R}_2[x]$ with respect to B. **[5]**

Q4) a) Let T be a linear operator on a finite dimensional vector space V and let $y \in V$. Define the minimal polynomial $m_T(x)$ of T and the annihilator $m_T^y(x)$ of y with respect to T. Prove that there exists a $u \in V$ such that $m_T(x) = m_T^u(x)$. **[6]**

b) Find the eigenvalues with multiplicities of the $n \times n$ matrix J whose all entries are 1. **[5]**

c) Let A be a 5×5 matrix on the field $\mathbb{Q}$ of rational numbers such that $A^7 = I_5$. Show that $A = I_5$. **[5]**

Q5) a) Let λ be an eigenvalue of an operator T on a vector space V. Define the geometric multiplicity of λ. prove that the geometric multiplicity of λ does not exceed its algebraic multiplicity. **[6]**

b) What do you mean by a diagonalizable matrix. Give two non-diagonal 3×3 matrices A and B such that A is diagonalizable but B is not diagonalizable. **[5]**

c) Give a 3×3 matrix A with real entries such that A is not triangulable over $\mathbb{R}$ but is triangulable on $\mathbb{C}$. **[5]**

Q6) a) Let T be a triangulable linear operator on a finite dimensional inner product space V. Prove that there exists an ordered orthonormal basis B of V such that the matrix of T with respect to B is upper triangular. **[6]**

b) Explain the rational canonical form of a matrix. Prove that two matrices are similar if and only if they have same rational canonical forms. **[5]**

c) Determine the Jordan canonical form of the matrix $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Hence deduce the geometric multiplicity of 1. **[5]**

Q7) a) State and prove Riesz representation theorem for finite dimensional vector spaces. **[6]**

b) Verify that the vectors (1, 2, 4, 1) and (–1, –2, 1, 1) are orthogonal with respect to standard inner product, and extend these to form an orthogonal basis. **[5]**

c) Show that for x, y in an inner product space V over $\mathbb{C}$:

$\|x+y\| = \|x\| + \|y\|$ if and only if $x = 0$ or $y = \lambda x$. for some $\lambda \in \mathbb{C}$. **[5]**

Q8) a) Define an adjoint of a linear map T : V $\to$ W. Prove the existence and uniqueness of the adjoint of T. **[6]**

b) Let A be an unitary matrix with integer entries. Prove that only non-zero entries in A are 1 and –1. Also show that there is exactly one nonzero entry in each row and in each column of A. **[5]**

c) Show that the eigenvalues of a real symmetric matrix are real. Also show that if A is a positive definite matrix, then A is invertible and A^{-1} is also positive definite. **[5]**

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Total No. of Questions :8]

SEAT No. :

P2052

[Total No. of Pages :2

[4821]-14

M.A. / M.Sc.

MATHEMATICS

MT-504: Number Theory

(2008 Pattern) (Semester - I)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Let p be a prime. Show that $x^2 \equiv -1 (\bmod p)$ has a solution if and only if p = 2 or $p \equiv 1 (\bmod 4)$. **[6]**

b) Find all integers that give remainder 1, 2, 3 when divided by 3, 4, 5 respectively. **[5]**

c) Prove that if p and q are district primes of the form $4k + 3$, and if $x^2 \equiv p \ (\bmod q)$ has no solution, then $x^2 \equiv q \ (\bmod p)$ has two solutions. **[5]**

Q2) a) State and prove Wilson's theorem. **[6]**

b) If g.c.d $(m, n) = 1$, then prove that $\phi(m\ n) = \phi(m)\ .\ \phi(n)$. **[5]**

c) Find all solutions of $101\,x + 99\,y = 437$. **[5]**

Q3) a) For any odd prime p, let $(a, p) = 1$, consider the integers $a, 2a, 3a ...... \left(\frac{p-1}{2}\right)a$ and their least positive residues modulo p. If n denotes the number of these residues that exceed $\frac{p}{2}$, then prove that $\left(\frac{a}{p}\right) = (-1)^n$. **[6]**

b) If an irreducible polynomial $p(x)$ divides $f(x).\ g(x)$, then prove that $p(x)$ divides at least one of the polynomials $f(x)$ and $g\ (x)$. **[5]**

c) Prove that $\sqrt{3} - 1$ and $\sqrt{3} + 1$ are associates in $Q\left(\sqrt{3}\right)$. **[5]**

P.T.O.

Q4) a) Let $f(n)$ be a multiplicative function and let $F(n)=\sum_{d|n} f(d)$. Prove that F(n) is multiplicative. **[6]**

b) Explain Pollard ρ method for factorisation. **[5]**

c) Find the highest power of 7 that divides 1000! **[5]**

Q5) a) If α and β are algebraic numbers, then prove that $\alpha + \beta$ and $\alpha\beta$ are algebraic numbers. **[8]**

b) prove that 3 is a quadratic residue of 13, but it is not quadratic residue of 7. **[5]**

c) Find a positive integer n such that $\mu(n) + \mu(n+1) + \mu(n+2) = 3$. **[3]**

Q6) a) State and prove the law of quadratic reciprocity. **[8]**

b) Find whether $x^3+2x-3\equiv 0\,(\text{mod}\, g)$ has solutions? If yes, then find solutions. **[5]**

c) Prove that $n^{12} - 1$ is divisible by 7 if $(n, 7) = 1$. **[3]**

Q7) a) Prove that, for every positive integer n, $\sum_{d|n}\phi(d)=n$. **[6]**

b) If α is any algebraic number, then prove that there is a rational integer b such that bα is an algebraic integer. **[5]**

c) Prove that, 3 is prime in Q(i), but not a prime in $Q\left(\sqrt{6}\right)$. **[5]**

Q8) a) Let a, b and $m > 0$ be given integers and let $g = (a, m)$. Then prove that $ax\equiv b\,(\text{mod}\, m)$ has a solution if and only if $g \mid b$. **[6]**

b) Find all real numbers x such taht

i) $[x] + [x] = [2x]$

ii) $\left[x+\frac{1}{2}\right]+\left[x-\frac{1}{2}\right]=[2x]$ **[5]**

c) Evaluate $\left(\frac{22}{105}\right)$ where $\left(\frac{22}{105}\right)$ is Jacobi's symbol. **[5]**

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Total No. of Questions :8]

SEAT No. :

P2053

[Total No. of Pages :3

[4821]-15

M.A. / M.Sc.

MATHEMATICS

MT-505: Ordinary Differential Equations

(2008 Pattern) (Semester - I)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) If $y_1(x)$ and $y_2(x)$ are two solutions of equation $y'' + P(x)y' + Q(x)y = 0$ on $[a, b]$, then prove that $y_1(x)$ and $y_2(x)$ are linearly dependent on this interval if and only if their Wronskian $W(y_1, y_2)$ is identically zero. **[8]**

b) Find the general solution of $y'' + 4y = 3\sin x$ by using method of undetermined coefficients. **[8]**

Q2) a) If $y_1(x)$ is one solution of the differential equation $y'' + P(x)y' + Q(x)y = 0$, then find the other solution. **[6]**

b) Find a particular solution of $y'' - 2y' - 3y = 64xe^{-x}$ by using method of variation of parameters. **[5]**

c) Verify that $y_1 = x$ is one solution of $x^2 y'' + 2x\,y' - 2y = 0$, find y_2 and general solution. **[5]**

Q3) a) State and prove sturm comparison theorem. **[8]**

b) Find the general solution of $\left(1 - x^2\right)y'' - 2xy' + p(p+1)y = 0$, about $x = 0$ by power series. **[8]**

P.T.O.

Q4) a) Let $u(x)$ be any non-trivial solution of $u'' + q(x)u = 0$ where $q(x) > 0$ for all $x > 0$. If $\int_1^\infty q(x)dx = \infty$, then prove that $u(x)$ has infinitely many zeros on the positive x - axis. **[8]**

b) Verify that origin is regular singular point and calculate two independent Frobenius series solution for the equation $4xy'' + 2y' + y = 0$. **[8]**

Q5) a) Find the general solution of the system **[8]**

$$\frac{dx}{dt} = x - 2y$$
$$\frac{dy}{dt} = 4x + 5y$$

b) Locate and classify the singular points on the x -axis of

$$x^2(x^2-1)^2 \; y'' - x(1-x)y' + 2y = 0.$$ **[4]**

c) Show that $y = C_1 e^x + C_2 e^{-x}$ is the general solution of the differential equation $y'' - y = 0$. **[4]**

Q6) a) Find the general solution near x = 0 of the hypergeometric equation $x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$, where a, b and c are constants. **[8]**

b) Determine the nature of the point $x = \infty$ for the equation $x^2y'' + xy' + (x^2-4)y = 0$. **[4]**

c) Find the nature and stability property of critical point (0, 0) for **[4]**

$$\frac{dx}{dt} = 5x + 2y$$
$$\frac{dy}{dt} = -17x - 5y.$$

Q7) a) If m_1 and m_2 are roots of the auxiliary of the system **[8]**

$$\frac{dx}{dt} = a_1x + b_1y$$
$$\frac{dy}{dt} = a_2x + b_2y$$

which are real, distinct and of same sign, then prove that the critical point (0, 0) is a node.

b) Show that the function $f(x, y) = xy^2$ satisfies Lipschitz condition on any rectangle $a \le x \le b$ and $c \le y \le d$; but it does not satisfy a Lipschitz condition on any strip $a \le x \le b$ and $-\infty < y < \infty$. **[8]**

Q8) a) Find the general solution of $(1-e^x)y'' + \frac{1}{2}y' + e^x y = 0$ near the singular point $x = 0$. **[8]**

b) Discuss the method of undetermined coefficients to find the solution of second order differential equation with constant coefficients. **[8]**

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Total No. of Questions :8]

SEAT No. :

P2054

[Total No. of Pages :2

[4821]-21

M.A / M.Sc.

MATHEMATICS

MT- 601: General Topology

(2008 Pattern) (Semester - II)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*

2) *Figures to the right indicate full marks.*

Q1) a) Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each x in U, there is an element C in $\mathcal{C}$ such that $x \in C \subseteq U$. Then prove that $\mathcal{C}$ is a basis for the topology of X. **[6]**

b) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X. Show that $\cap \mathcal{T}_\alpha$ is a topology on X. Is $\cup \mathcal{T}_\alpha$ a topology on X? Justify. **[6]**

c) If X is any set, then prove that the collection of all one- point subsets of X is a basis for the discrete topology on X. **[4]**

Q2) a) Define usual topology and lower limit topology on $\mathbb{R}$. Establish a relation among them. **[6]**

b) Define a subspace topology. If A is a subspace of $(X,\mathcal{T})$ and B is a subspace of $(Y,\mathcal{T}')$, then show that the product topology on A x B is same as the topology A x B inherits as a subspace of X x Y. **[6]**

c) Let Y = [–1, 1], which of the following sets are open in Y? Open in $\mathbb{R}$? Justify. **[4]**

i) $A = \left\{ x \mid \frac{1}{2} \le |x| < 1 \right\}$

ii) $B = \left\{ x \mid \frac{1}{2} < |x| < 1 \right\}$

P.T.O.

Q3) a) Let Y be subspace of X and A be a subset of Y. Let A denote the closure of A in X then show that the closure of A in Y is $\bar{A} \cap Y$. **[5]**

b) Show that every finite set is closed in a Hausdorff space. **[5]**

c) Show that every order topology is Hausdorff. **[6]**

Q4) a) Let X be a space satisfying T_1 axiom and A be a subset of X. Prove that the point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A. **[6]**

b) State and prove pasting lemma. **[6]**

c) Define box topology and product topology. What is the relation between them? **[4]**

Q5) a) Define quotient topology and given an example of a quotient map which is not a closed map? **[5]**

b) Show that the union of a collection of connected subspaces that have a common point is connected. **[5]**

c) For locally path connected space X, show that every connected open set is path connected. **[6]**

Q6) a) Show that every compact subspace of a Hausdorff space is closed. **[6]**

b) Show that in the finite complement topology on $\mathbb{R}$ every subspace is compact. **[5]**

c) Justify whether true or false: A topological space X is compact if and only if X is limit point compact. **[5]**

Q7) a) Give an example of a Hausdorff space which is not regular. **[5]**

b) Prove that every compact Hausdorff space is normal. **[6]**

c) Let X be locally compact Hausdorff space and $A \subseteq X$. If A is closed in X or open in X, show that A is locally compact. **[5]**

Q8) a) State and prove Tychonoff theorem. **[12]**

b) State Urysohn lemma. **[2]**

c) Define a regular space and give its example. **[2]**

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Total No. of Questions :8]

SEAT No. :

P2055

[Total No. of Pages :2

[4821]-22

M.A / M.Sc.

MATHEMATICS

MT- 602: Differential Geometry

(2008 Pattern) (Old Pattern) (Semester - II)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to right indicate full marks.*

Q1) a) Show that n-sphere is n-surface in $\mathbb{R}^{n+1}$. **[4]**

b) State and prove Lagranges multiplier theorem. **[8]**

c) Sketch the level set and graph of the function $f(x, y) = -x^2 + y^2$. **[4]**

Q2) a) Show that set of vectors tangent to f^{-1}(c) at point p, where p is regular point for the smooth function $f : U \to \mathbb{R}$ is equal to $[\nabla f(p)]^{+}$. **[8]**

b) Find the gradient field for the vector field $f(x_1, x_2) = x_1^2 + x_2^2$. **[4]**

c) Is mobius band a 2-surface? Why? **[4]**

Q3) a) Show that Gauss map maps a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ onto unit sphere S^n. **[12]**

b) Sketch the vector field on $\mathbb{R}^2$ for $\bar{X}(P) = (P, X(P))$ where X(P) = (0, 1). **[4]**

Q4) a) Find velocity, acceleration and the speed of the parametrized curve given by $\alpha(t) = (\cos 3t, \sin 3t)$. **[6]**

b) Show that there exists a maximal integral curve passing through given point for the smooth tangent vector field on n-surface. **[10]**

Q5) a) Define Levi - civita parallelism along with its properties. **[5]**

b) Show that a parametrized curve α in the unit n-sphere $x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1$ is geodesic iff it is of the form $\alpha(t) = (\cos at)\, e_1 + (\sin at) e_2$ for some orthogonal pair of unit vectors $\{e_1, e_2\}$ in $\mathbb{R}^{n+1}$ and some $a \in \mathbb{R}$. **[6]**

c) Compute $\nabla_V f$ where $f : \mathbb{R}^{n+1} \to \mathbb{R}, n = 2, v \in \mathbb{R}_p^3, p \in \mathbb{R}^3$ given by $f(x_1, x_2, x_3) = x_1 x_2 x_3^2$ and $\bar{v} = (1,1,1,a,b,c)$. **[5]**

Q6) a) Show that the Weingarten map Lp is self adjoint. **[8]**

b) Compute the curvature of a circle in $\mathbb{R}^2$ of radius 4 passing through (5, 6) oriented by outward normal $\nabla f / \|\nabla f\|$. **[6]**

c) State second derivative test for the local maxima of the smooth function. **[2]**

Q7) a) Show that there exists a global parametrization of oriented plane curve C iff C is connected. **[12]**

b) Calculate the Gaussian curvature of the ellipsoid $S : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$ oriented by its outward normal. **[4]**

Q8) a) Show that on each compact oriented n-surface S in $\mathbb{R}^{n+1}$, there exists a point p such that the second fundamental form p is definite. **[8]**

b) Compute the Weingarten map for the hyperplane $a_1 x_1 + a_2 x_2 + ... + a_{n+1} x_{n+1} = b$ $[(a_1, a_2, ..., a_{n+1}) \neq (0, 0, ..., 0)]$ **[6]**

c) State inverse function theorem for n-surfaces. **[2]**

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Total No. of Questions :8]

SEAT No. :

P2056

[Total No. of Pages :2

[4821]-23

M.A. / M.Sc.

MATHEMATICS

MT-603: Groups and Rings

(2008 Pattern) (Semester - II)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Prove that the set of all 2×2 matrices with entries from $\mathbb{R}$ and determinant 1 is a group under matrix multiplication. **[5]**

b) Define a factor group. Give an example of a group which is non abelian but whose factro group is abelian. Justify the answer. **[5]**

c) Prove that a group G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$. **[6]**

Q2) a) Prove that $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ are not isomorphic. **[5]**

b) Give an example of an infinite group whose every subgroup is of finite order. Justify the answer. **[5]**

c) List all 6 cyclic subgroups of in U(15). **[6]**

Q3) a) Prove that every permutation of a finite set can be written as a cycle or a product of disjoint cycles. **[5]**

b) Show that the group $\mathbb{R}^{+}$ under multiplication is isomorphic to $\mathbb{R}$ under addition. **[5]**

c) Find $Aut(\mathbb{Z}_6)$, the group of automorphisms of $\mathbb{Z}_6$. **[6]**

P.T.O.

Q4) a) Find the inverse and the order of each of the following permutations in S_{11} **[5]**

i) (1 3 4 6) (2 5 7)

ii) (1 2 10 4) (3 10) (7 8 11)

b) Let G be a group and Z(G) be a center of G. If G/Z(G) is cyclic, then prove that G is abelian. **[5]**

c) Let G be a finite abelian group of order n and p be prime dividing n. Then prove that G has an element of order p. **[6]**

Q5) a) State and prove the Orbit-Stabilizer Theorem. **[5]**

b) Give an example of a non abelian group whose all proper subgroups are cyclic. **[5]**

c) Let A(n) be the set of all even permutations on an n element set. Prove that A(n) is a subgroup of S_n and order of A(n) = $n!/2$. **[6]**

Q6) a) State and prove the Cayley's Theorem **[5]**

b) Determine all the homomorphisms from $\mathbb{Z}_{15}$ to $\mathbb{Z}_{25}$. **[5]**

c) Find all the non isomorphic abelian group of order1800. **[6]**

Q7) a) Determine all the groups of order 99. **[5]**

b) Prove that a group of order 375 has a subgroup of order 15. **[5]**

c) If |G| is a group of order pq where p and q are primes, $p < q$, and p does not divide $q - 1$, then prove that G is cyclic. **[6]**

Q8) a) Prove that the group of order 100 is not simple. **[5]**

b) Define center of a group. Find the center of S_3. **[5]**

c) State and prove the Lagrange's theorem for finite groups. Give an example where the converse of Lagrange's theorem is not true. **[6]**

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Total No. of Questions :8]

SEAT No. :

P2057

[Total No. of Pages :3

[4821]-24

M.A. / M.Sc.

MATHEMATICS

MT-604: Complex Analysis

(2008 Pattern) (Old Course) (Semester - II)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Let $\sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R > 0. Prove that the radius of convergence of this series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ is also R. **[6]**

b) Let G be either the whole plane $\mathbb{C}$ or some open disk. If $u : G \to \mathbb{R}$ is a harmonic function then prove that u has a harmonic conjugate. **[5]**

c) Let G be a region and define $G^* = \{z \mid \overline{z} \in G\}$. If $f : G \to \mathbb{C}$ is analytic, prove that $f^* : G^* \to \mathbb{C}$ defined by $f^* : (z) = f(\overline{z})$, is also analytic. **[5]**

Q2) a) Define Mobius transformaton and prove that every Mobius transformation maps circles of $\mathbb{C}_\infty$ onto circles of $\mathbb{C}_\infty$. **[6]**

b) Let G and Ω be open subset of $\mathbb{C}$. Let $f : G \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ are continuous function such that $f(G) C \Omega$ and $g(f(z)) = z \; \forall z \in G$. If g is differentiable and $g'(z) \neq 0$ then prove that f is differentiable and $f'(z) = \frac{1}{g'(f(z))}$. **[7]**

c) Let $f(z) = |z|^2 = x^2 + y^2$ for $z = x + iy \in \mathbb{C}$. Show that f is continuous. **[3]**

P.T.O.

Q3) a) If $f:G\to\mathbb{C}$ is analytic then prove that *f* preserves angles at each point z_o of G where $f'(z_o)\neq 0$. **[6]**

b) Let f be analytic in the disk B(*a;*R) and suppose that γ is a closed rectifiable curve in B(*a;*R). Prove that $\int_r t=0$. **[6]**

c) Let $\gamma(t)=e^{it}$ for $0\le t\le 2\pi$. Find $\int_r z^n dz$ for every integer *n.* **[4]**

Q4) a) Let f be analytic in open ball B(*a;*R) prove that $f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n$ for $|z-a|<R,\ a_n=\dfrac{f^{(n)}(a)}{n!}$ and this series has radius of convergence $\geq R$. **[6]**

b) If *p*(*z*) is a nonconstant polynomial then prove that there is a complex no; *a* with *p*(*a*) = 0. **[5]**

c) Let *f*(*z*) be a polynomial of degree n and let R > 0 be sufficiently large so that *f* never vanishes in $\{z:|z|>R\}$. If $\gamma(t)=\text{Re}^{it}$ $0\le t\le 2\pi$, show that $\int_r \dfrac{f'(z)}{f(z)}dz=2\pi$ in. **[5]**

Q5) a) Prove that if G is a region and let $f:G\to\mathbb{C}$ be a continuous function such that $\int_T f=0$ for every triangular path T in G then *f* is analytic in G. **[6]**

b) If $\gamma:[0,1]\to\mathbb{C}$ is a closed rectifiable curve and $a\notin\{\gamma\}$ then prove that $\dfrac{1}{2\pi i}\int_\gamma \dfrac{dz}{z-a}$ is an integer. **[5]**

c) Let *f* be analytic on D = B(0 ; 1) and suppose $|f(z)|\le|$ for $|z|<|$. Show that $|f'(0)|\le|$. **[5]**

Q6) a) If *f* has an essential singularity at *z* = *a*, then prove that, for every *r* > 0 the closure of the set *f* [*ann* (*a* ; 0, *r*)] is equal to $\mathbb{C}$. **[6]**

b) If G is simply connected and $f:G\to\mathbb{C}$ is analytic then prove that *f* has a primitive in G. **[6]**

c) Evaluate $\int_r \dfrac{2z+1}{z^2+z+1}dz$ where γ is the circle $|z|=2$. **[4]**

Q7) a) State and prove Residue theorem. **[8]**

b) Prove that $\int_0^{\pi}\frac{d\theta}{a+\cos\theta}=\frac{\pi}{\sqrt{a^2-1}}$ where a > 1. **[6]**

c) State Rouche's theorem. **[2]**

Q8) a) Let G be a region in $\mathbb{C}$ and f an analytic function on G. Suppose there is a constant M such that $\limsup_{z\to a}|f(z)|\le M$ for all a in $\partial_\infty G$. Prove that $|f(z)|\le M$ for all z in G. **[6]**

b) If | a | < | then prove that the mobius transformation $\varphi_a(z)=\frac{z-a}{1-\bar{a}z}$ is a one-one map of D = {z : | z | < |} onto itself; the inverse of φ_a is φ_{-a}. Also prove φ_a maps ∂D on to ∂D ϕ_a (a) = 0, ϕ_a (0) = 1 – | a |^2 and $\varphi_a'(a)=(|-|a|^2)^{-1}$. **[5]**

c) Evaluate the integral $\int_r \frac{e^z-e^{-z}}{z^n}dz$ where n is a positive integer and $\gamma(t)=e^{it}\ \ 0\le t\le 2\pi$. **[5]**

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Total No. of Questions :8]

SEAT No. :

P2058

[Total No. of Pages :2

[4821]-25

M.A / M.Sc.

MATHEMATICS

MT- 605: Partial Differential Equations

(2008 Pattern) (Semester - II)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Find the general solution of $yzp + xzq = xy$. **[3]**

b) Eliminate the arbitrary function F from $F(x+y, x-\sqrt{z})=0$. **[3]**

c) State the condition for the equations $f(x, y, z, p, q) = 0$ and $g(x, y, z, p, q) = 0$. **[3]**

d) Prove that the pfaffian differential equation

$\bar{X}.d\bar{r} = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0$ is integrable if and only if $\bar{X}.curl\ \bar{X} = 0$. **[7]**

Q2) a) Verify that the equation $zydx + xzdy + xydz = 0$ is integrable and find its primitives. **[6]**

b) Find the complete integral of $p = (z + qy)^2$ by using Charpit's method. **[6]**

c) Solve the nonlinear partial differential equation $zpq - p - q = 0$ **[4]**

Q3) a) Explain the method of solving the following first order partial differential equations: **[8]**

i) $f(z, p, q) = 0$

ii) $g(x, p) = h(y, q)$

b) Find the integral surface of the equation $x^3 p + y(3x^2 + y)q = z(2x^2 + y)$ which passes through the curve C: $x_0 = 1, y_0 = S, z_0 = S(1+S)$. **[8]**

P.T.O.

Q4) a) Explain Jacobi's method for solving the partial differential equation $f(x, y, z, p, q) = 0$ and find a complete integral of the equation $(p^2x + q^2y) - z = 0$. **[10]**

b) For a nonlinear first order PDE, $f(x, y, z, p, q) = 0$ derive analytic expression for the Monge Cone at (x_0, y_0, z_0). Further consider the equation $p^2 + q^2 = 1$. Find the equation of the Monge Cone with vertex at (0, 0, 0) **[6]**

Q5) a) If an element $(x_0, y_0, z_0, p_0, q_0)$ is common to both an integral surface $z = z(x, y)$ and a characteristic strip, then show that the corresponding characteristic curve lies completely on the surface. **[8]**

b) Reduce the equation $y^2 u_{xx} - 2xy\, u_{xy} + x^2 u_{yy} = \frac{y'^2}{x} u_x + \frac{x^2}{y} u_y$, into a canonical form and solve it. **[8]**

Q6) a) State Dirichlet problem for rectangle and find it's solution. **[8]**

b) State and prove Kelvin's inversion theorem. **[8]**

Q7) a) Suppose that $u(x, y)$ is harmonic in a bounded domain D and continuous in $\bar{D} = D \bigcup B$ then prove that u attain its maximum on the boundary B of D. **[8]**

b) Prove that the solution of following problem exist then it is unique **[8]**

$$u_{tt} - c^2 u_{xx} = F(u,t), 0 < x < l, t > 0$$
$$u(x,0) = f(x), 0 \le x \le l$$
$$u_t(x,0) = g(x)$$
$$u(0,t) = u(l,t) = 0, t \ge 0$$

Q8) a) Solve the Quasi - Linear equation $zz_u + z_y = 1$ containing the initial data curve $x_0 = s, y_0 = s$ $z_0 = \frac{1}{2}s$ for $0 \le s \le 1$. **[4]**

b) Using Duhamel's principle find the solution of non homogeneous equation $\begin{array}{l} u_{tt} - c^2 u_{xx} = f(x,t), -\infty < x < \infty, t > 0 \\ u(x,0) = u_t(x,0) = 0, -\infty < x < \infty \end{array}$ **[6]**

c) Using the variable separable mehod solve $u_t = ku_{xx}; 0 < x < a, t > 0$ which satisfies condition $u(0, t) = u(a, t) = 0$; $t > 0$ and $u(x, 0) = x(a - x)$; $0 \le x \le a$. **[6]**

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Total No. of Questions : 4]

SEAT No. :

P2059

[Total No. of Pages :2

[4821]-26

M.A./M.Sc.

MATHEMATICS

MT:606- Object Oriented Programming with C^{++}

(2008 Pattern) (Semester-II)

Time : 2 Hours] *[Max. Marks :50*

Instructions to the candidates:

1) *Figures to the right side indicate full marks.*
2) *Question one is compulsory.*
3) *Attempt any two questions from Q.2, Q.3 and Q.4.*

Q1) Attempt any Ten of the following: **[20]**

a) Write syntax for accessing array element.

b) Write output of following program.

```
#   include <iostream.h>
    int main( )
    {
     cout << "mathematics is beautiful";
     return o;
     }
```

c) What are C^{++} keywords? Give four examples.

d) What is use of scope resolution operator?

e) What is data encapsulation?

f) Define hybrid inheritance.

g) Write the syntax of friend functions.

h) Explain the term "message passing".

i) State one difference between break and continue.

j) Write a program to multiply and divide two real numbers a=2.5 and b=1.5 using inline function.

P.T.O.

k) What is operator overloading?

l) What are disadvantages of macros?

Q2) a) Write an object oriented program in C^{++} to multiply two matrices. Let M_1 and M_2 be two matrices. Find out $M_3=M_1 * M_2$. **[10]**

b) Compare dynamic memory management in C and C^{++}. **[5]**

Q3) a) Write a program to perform the addition time in the hour and minutes format . **[10]**

b) Write a note on general form of class declaration. **[5]**

Q4) a) Define **[9]**

i) Call by value.

ii) Call by reference.

iii) Return by reference with examples.

b) Write benifit of object oriented programming. **[6]**

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Total No. of Questions : 8]

SEAT No. :

P2060

[Total No. of Pages :2

[4821]-31
M.A./M.Sc.
MATHEMATICS
MT:701- Functional Analysis
(2008 Pattern) (Semester-III)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Let H be a Hilbert space and f be a functional on H. Prove that there exits a unique vector y in H such that $f(x) = < x, y >$ for every $x \in \text{H}$. **[8]**

b) Let X be a normed space over $\mathbb{C}$. Let $0 \neq a \in \text{X}$. Show that there is some functional f on X such that $f(a) = \|a\|$ and $\|f\| = 1$. **[6]**

c) Show that the norm of an isometry is 1 . **[2]**

Q2) a) Show that an operator T on a finite dimensional Hilbert space is normal if and only if its adjoint T* is a polynomial in T. **[6]**

b) State and prove Hahn-Banach Theorem. **[8]**

c) Let H be 2-dimensional Hilbert space. Let the operator T on H be defined by $Te_1=e_2$ and $Te_2= -e_1$. Find the spectrum of T. **[2]**

Q3) a) If T is an operator on a Hilbert space H, then prove that T is normal if and only if its real and imaginary parts commute. **[6]**

b) i) Let X and Y be normed spaces. If X is finite dimensional then show that every linear transformation from X to Y is continuous. **[4]**

ii) Give an example of a discontinuous linear transformation. **[4]**

c) A linear operator $\text{T} : l^2 \to l^2$ is defined by $T(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...)$. Find T*. **[2]**

P.T.O.

Q4) a) Show that a closed convex subset C of a Hilbert space H contains a unique vector of samllest norm. **[6]**

b) If T is any operator on a Hilbert space H and if α, β are scalars such that $|\alpha| = |\beta|$, then show that $\alpha T + \beta T^*$ is normal. **[4]**

c) If T is an operator on a Hilbert space H for which $<Tx,x>=0$ for all $x \in H$, then prove that T=0. **[6]**

Q5) a) Prove that $\|T^*\| = \|T\|$ and $\|T^*T\| = \|T\|^2$. **[8]**

b) Give an example of a (non-identity) operator which is self-adjoint. Justify. **[4]**

c) Show that the unitary operators on a Hilbert space H form a group. **[4]**

Q6) a) State and prove the uniform Boundedness principle. **[8]**

b) Let T be an operator on H. If T is non-singular, then show that $\lambda \in \sigma(T)$ if and only if $\lambda^{-1} \in \sigma(T^{-1})$. **[4]**

c) Show that every positive operator on a finite dimensional Hilbert space has a unique positive square root. **[4]**

Q7) a) Let T be a normal operator on H, with spectrum $\{\lambda_1, \lambda_2, ..., \lambda_m\}$. Show that T is self-adjoint if and only if each λi is real . **[4]**

b) Let M be a closed linear subspace of a normed linear space N and T be a natural mapping of N onto N/M defined by $T(x)=x+M$. Show that T is a bounded linear transformation with $\|T\| \leq 1$. **[4]**

c) If {ei} is an orthonormal set in a Hilbert space H and x is any vector in H, then prove that the set $S = \{ei : < x, ei > \neq 0\}$ is either empty or countable. **[8]**

Q8) a) State and prove the closed graph Theorem. **[8]**

b) Let $X=C'[a,b]$ with norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ and $Y + C'[a,b]$ with sup norm. Let $F: X \rightarrow Y$ be defined as identity map. Show that F is continuous but F^{-1} is discontinuous. Why the open mapping theorem fails? Justify. **[8]**

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Total No. of Questions : 8]

SEAT No. :

P2061

[Total No. of Pages :2

[4821]-32
M.A./M.Sc.
MATHEMATICS
MT-702: Ring Theory
(2008 Pattern) (Semester-III)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Prove that any finite integral domain is a field. **[6]**

b) Show that if $n = a^k b$ for some integers a and b then $\overline{ab}$ is a nilpotent element of $\frac{Z}{nZ}$. **[5]**

c) If x is a nilpotent element of the commutative ring R then prove that $1+x$ is unit in R. **[5]**

Q2) a) If A is a subring and B is an ideal of the ring R then prove that $\frac{A+B}{B} \cong \frac{A}{A \cap B}$. **[6]**

b) Prove or disprove **[6]**

The ideal $I = (2, x)$, generated by 2 and x in $Z[x]$ is a principal ideal.

c) If $x^2 + x + 1$ is an element of $E = F_2[x]$ then find $\bar{E} = \frac{F_2[x]}{(x^2 + x + 1)}$. **[4]**

Q3) a) Prove that in a Boolean ring the ideal I is prime ideal if and only if it is maximal ideal. **[6]**

b) If R is a commutative ring and if every element of R is either nilpotent or a unit then prove that R has exactly one prime ideal. **[5]**

c) If F is a field then prove that F contains a unique smallest subfield F_0 which is isomorphic to either Q or Z_p for some prime p. **[5]**

Q4) a) If R is a commutative ring with unity and A_1, A_2 are comaximal ideals in R then prove that $\frac{R}{A_1A_2} \simeq \frac{R}{A_1 \cap A_2} \simeq \frac{R}{A_1} \times \frac{R}{A_2}$. **[8]**

b) If R is the ring of all continuous functions on [0,1] and I is the collection of functions $f(x)$ in R with $f\left(\frac{1}{3}\right) = f\left(\frac{1}{2}\right) = 0$ then prove that I is an ideal of R but not a prime ideal of R. **[8]**

Q5) a) Define Euclidean domain. Prove that every ideal in a Euclidean domain is principal. **[6]**

b) Prove that the quotient ring $\frac{Z[i]}{I}$ is finite for any non-zero ideal I of $Z[i]$. **[6]**

c) Find a generator for the ideal $I = (2+3i, 4+7i)$ in $Z[i]$. **[4]**

Q6) a) Prove that every non-zero prime ideal in a principal ideal domain is a maximal ideal. **[6]**

b) If R is an integral domain and $R[x]$ is a PID then prove that R is a field. Use above result to show that neither $Z[x]$ nor $R[x,y]$ are PIDs. **[6]**

c) Show that the ring $R = \left\{\frac{m}{n} \middle| m, n \in Z, n \text{ is odd}\right\}$ is a principal ideal domain. **[4]**

Q7) a) Prove that every irreducible element in UFD is a prime element. **[6]**

b) Prove that every Noetherian ring wihich is also an integral domain is a factorisation domain. **[6]**

c) Give an example to show that in a UFD the g.c.d of two elements a and b need not be expressible in the form $\lambda a + \mu b$ for some $\lambda, \mu \in R$. **[4]**

Q8) a) If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \text{------} + a_0$ is a polynomial of degree n with integer coefficients. If $\frac{r}{s} \in Q$ is in lowest terms with $p\left(\frac{r}{s}\right) = 0$ then prove that r divides a_0 and s divides a_n. **[6]**

b) State and prove Eisenstein's criterion for $Z[x]$. **[6]**

c) Prove that the polynomial $x^4 + 4x^3 + 6x^2 + 2x + 1$ is irreducible over Z. **[4]**

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Total No. of Questions : 8]

SEAT No. :

P2062

[Total No. of Pages :2

[4821]-33
M.A./M.Sc.
MATHEMATICS
MT:703- Mechanics
(2008 Pattern) (Semester-III)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Explain the following terms. **[8]**

i) D' Alembert's principle.

ii) Virtual displacement.

iii) Lagrange's equations for Non-holonomic constraints.

iv) Eulerian angles.

b) Obtain Lagrange's equation of motion from D' Alembert's principle. **[8]**

Q2) a) A particle of mass m moves in a plane under the action of a conservative force F with components $F_x = -k^2(2x + y), F_y = -k^2(x + 2y),\ k$ is constant . Find the total energy of the motion the Lagrangian and the equation of motion of particle. **[8]**

b) Find the Lagrangian of a particle moving in the field of force given by

$$F = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2r\ddot{r}}{c^2}\right).$$ **[8]**

Q3) a) Find the Lagrange's equation of motion for simple pendulum. **[6]**

b) Show that the generalized momentum corresponding to a cyclic co-ordinate is conserved. **[4]**

c) Show that the geodesic in a Euclidean plane is a straight line. **[6]**

Q4) a) Find the Euler-Lagrange differential equations satisfied by twice differentiable function y(x) which extremizes the functional $I(y(x))=\int_{x_1}^{x_2} f(x,y,y')dx.$, where y is prescribed at the end points. **[8]**

b) State and prove the Brachistochrone problem in calculus of variations. **[8]**

Q5) a) Deduce Newton's second law of motion from Hamilton's principle. **[6]**

b) Obtain Hamilton's equation of motion from the Hamilton's Principle. **[8]**

c) Express Hamilton's canonical equations of motion in terms of poisson brackets. **[2]**

Q6) a) Find the Routhian for the Lagrangian

$$L=\frac{1}{2}l_3(\dot{\psi}+\dot{\phi}\cos\theta)^2+\frac{1}{2}l_1(\dot{\theta}+\dot{\phi}^2\sin^2\theta)-\text{mgl}\cos\theta.$$ **[6]**

b) If $f=f(q_j,\dot{q}_j,t)$ then show that $\Delta f=\delta f+\Delta t.\frac{df}{dt}$. **[6]**

c) Prove that central force motion is always motion in a plane. **[4]**

Q7) a) Prove the Keple's third law of planetary motion. **[8]**

b) Prove that an orthogonal transformation in the inverse matrix is identified by the transpose of the matrix. **[8]**

Q8) a) Eulerian angles $\phi=\frac{\pi}{4},\theta=\frac{\pi}{2}$ and $\psi=\frac{\pi}{4}$ bring space frame s into coincidence with body frame S' with common origin. Find the transfor mation relating co-ordinates of S and S'. Show that if the transformation is equivalent to a rotation through an angle x about some axis through origin them $\cos\left(\frac{x}{2}\right)=\frac{1}{2}$. **[6]**

b) Show that the transformation $p=\frac{1}{Q},q=PQ^2$ is canonical and find the generating function. **[6]**

c) Prove that Poisson brackets are invariant under canonical transformation. **[4]**

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Total No. of Questions : 8]

SEAT No. :

P2063

[Total No. of Pages :3

[4821]-34

M.A./M.Sc.

MATHEMATICS

MT-704: Measure and Integration

(2008 Pattern) (Semester-III)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right side indicate full marks.*
3) *All symbols have their usual meanings.*

Q1) a) Show that every non-empty open set G in $\mathbb{R}$ is a union of countably many disjoint open intervals. **[6]**

b) If $E_i \in \mathcal{B}$ then prove that $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mu E_i$. **[5]**

c) Let $(X,\mathcal{B},\mu)$ be a measure space and $Y \in \mathcal{B}$. Let $\mathcal{B}_Y$ consist of those sets of $\mathcal{B}$ that are contained in Y and $\mu_Y(E) = \mu(E)$ if $E \in \mathcal{B}_Y$. Then show that $(Y,\mathcal{B}_Y,\mu_Y)$ is a measure space. **[5]**

Q2) a) Define a σ-algebra. Show that the class of measurable sets $\mathcal{M}$ is a σ-algebra. **[6]**

b) Let *c* be any real number and let *f* and *g* be real valued measurable functions defined on the same measurable set E. Show that $f+c, cf, f+g, f-g$ and *fg* are measurable. **[6]**

c) For $k>0$ and $A \subseteq R$ let $kA = (x : k^{-1}x \in A)$. Show that **[4]**

i) $m^*(kA) = km^*(A)$,

ii) A is measurable iff *k*A is measurable.

Q3) a) Let for each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathcal{B}$ such that $\mu(B_\alpha - B_\beta)=0$ for $\alpha < \beta$. Then prove that there is a measurable function *f* such that $f \le \alpha$ a.e. on B_α and $f \ge \alpha$ a.e. on $X \sim B_\alpha$. **[6]**

P.T.O.

b) State and prove Fatou's Lemma. **[6]**

c) Show that the set of numbers in [0,1] which possess decimal expansions not containing the digit 5 has measure zero. **[4]**

Q4) a) Let v be a signed measure on the measurable space $(X,\mathscr{B})$ then prove that there is a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \phi$. **[6]**

b) i) Define signed measure. **[6]**

ii) Let v be a signed measure on $(X,\mathscr{B})$ and $E \in \mathscr{B}$ with $v(E)>0$. Then show that there exits A, a set positive with respect to v, such that $A \subset E$ and $v(A)>0$.

c) Show that $\mu(E_1 \Delta E_2) = 0$ implies $\mu E_1 = \mu E_2$ provided that $E_1, E_2 \in \mathscr{B}$. **[4]**

Q5) a) Let $(X,\mathscr{B},\mu)$ be a σ-finite measure space and let v be a measure defined on $\mathscr{B}$ which is absolutely continuous with respect to μ. Then prove that there is a nonnegative measurable function f such that for each set E in $\mathscr{B}$ we have $vE = \int_E f d\mu$. **[6]**

b) Let $(X, \mathscr{G}, \mu)$ and $(Y, \mathscr{B}, v)$ be two complete measure spaces and f an integrable function on X×Y. Then prove the following: **[6]**

i) For almost all x the function f_x defined by $f_x(y)=f(x,y)$ is an integrable function on Y.

ii) For almost all y the function f_x defined by $f^y(x) = f(x,y)$ is an integrable function on X.

iii) $\int_Y f(x,y)dv(y)$ is an integrable function on X.

iv) $\int_X f(x,y)d\mu(y)$ is an integrable function on Y.

c) Give an example of a function such that $|f|$ is measurable but f is not. **[4]**

Q6) a) Let F be a bounded linear functional on $L^p(\mu)$ with $1 \le p < \infty$ and μ a σ-finite measure. Then show that there is a unique element g in L^q where $1/p+1/q=1$, such that $F(f) = \int fg d\mu$ with $\|F\| = \|g\|_q$. **[6]**

b) Show that the class $\mathcal{B}$ of μ^*-measurable sets is a σ-algebra. **[6]**

c) An arbitrary set $E \subset X$ is μ^*-measurable if and only if $\mu^* O \ge \mu^*(O \cap E) + \mu^*(O \cap \tilde{E})$ for each open O with $\mu^* O < \infty$. **[4]**

Q7) a) i) Define product measure. **[8]**

ii) Let E (subset of X×Y) a set in $\mathcal{R}_{\sigma\delta}$ and x be a point of X. Then show that E_x(x cross section E) is a measurable subset of Y.

b) Let E be a set in $\mathcal{R}_{\sigma\delta}$ with $\mu \times v(E) < \infty$. Then show that the function g defined by $g(x) = \mu E_x$ is a measurable function of x and $\int g d\mu = \mu \times v(E)$. **[8]**

Q8) a) Let μ be a measure on an algebra $\mathcal{G}$ and μ^* the outer measure induced by μ. Then prove that the restriction $\bar{\mu}$ of μ^* to the μ^*-measurable sets is an extension of μ to σ-algebra containing $\mathcal{G}$. **[6]**

b) Let μ be a finite measure defined on σ-algebra which contains all the Baire sets of a locally compact space X. If μ is inner regular then show that it is regular. **[6]**

c) Let B be a μ^*-measurable set with $\mu^* B < \infty$ then prove that $\mu_* B = \mu^* B$. **[4]**

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Total No. of Questions : 8]

SEAT No. :

P2064

[Total No. of Pages :2

[4821]-35

M.A./M.Sc.

MATHEMATICS

MT-705: Graph Theory

(2008 Pattern) (Semester-III)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Prove that an edge is a cut edge if and only if it belongs to no cycle. **[6]**

b) Prove that the isomorphism relation is an equivalence relation on the set of simple graphs. **[6]**

c) Prove that the Petersen graph has girth 5. **[4]**

Q2) a) Prove that K_n decomposes into three pairwise–isomorphic subgraphs if and only if $n+1$ is not divisible by 3. **[8]**

b) Prove that a graph G is Eulerian if it has at most one nontrivial component and its vertices all have even degree. **[8]**

Q3) a) Show that the number of vertices in a self-complementary graph is either $4k$ or $4k+1$, where k is a positive integer. **[6]**

b) Prove that for a connected nontrivial graph with exactly $2k$ odd vertices, the minimum number of trails that decompose it is $max\{k,1\}$. **[6]**

c) Show that a hypercube Q_k is k-regular and $e(Q_k)=k2^{k-1}$. **[4]**

Q4) a) If T is a tree with k edges and G is a simple graph with $\delta(G) \geq k$, then T is a subgraph of G. **[7]**

b) Use Cayley's Formula to prove that the graph obtained from K_n by deleting an edge has $(n-2)n^{n-3}$ spanning trees. **[6]**

c) Show that every graph has an even number of vertices of odd degree. **[3]**

P.T.O.

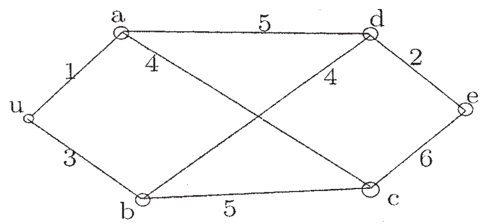
Q5) a) State and prove the Havel-Hakimi Theorem. **[10]**

b) Prove that for $k>0$, every k-regular bipartite graph has a perfect matching. **[6]**

Q6) a) Prove that two blocks in a graph share at most one vertex. **[6]**

b) Prove that if G is a simple graph with *diam* $G \geq 3$, then *diam* $\bar{G} \leq 3$. **[5]**

c) Using Dijkstra's algorithm find the shortest distance from u to every other vertex in the following graph. **[5]**



Q7) a) Prove that in a connected weighted graph G, Kruskal's Algorithm constructs a minimum-weight spanning tree. **[6]**

b) Let $\alpha'(G)$, $\beta'(G)$ and $n(G)$ denotes maximum size of matching, minimum size of edge cover and number of vertices in G respectively. Prove that if G is a graph without isolated vertices, then $\alpha'(G) + \beta'(G) = n(G)$. **[10]**

Q8) a) Prove that every component of the symmetric difference of two matchings is a path or an even. **[5]**

b) Explain the Ford-Fulkerson labeling algorithm to find an f-augmenting path. **[5]**

c) Prove that if x and y are distinct vertices of a graph G, then the minimum size of an x,y - disconnecting set of edges equals the maximum number of pairwise edge-disjoint x,y - paths. **[6]**

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Total No. of Questions : 8]

SEAT No. :

P2065

[Total No. of Pages :2

[4821]-41
M.A./M.Sc.
MATHEMATICS
MT-801: Field Theory
(2008 Pattern) (Semester-IV) (Old)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *All questions carry equal marks.*
3) *Figures to the right indicate full marks.*

Q1) a) If $f(x) = a_0 + a_1 x + \text{-----} + a_{n-1}x^{n-1} + x^n \in Z[x]$ is monic polynomial and if $f(x)$ has a root $a \in Q$, then prove that $a \in Z$ and a divides a_0. **[6]**

b) Is $f(x) = x^4 - 2$ irreducible over the ring of Gaussian integers? Why? **[6]**

c) Let F be a finite field. Construct a polynomial over F which has none of the elements of F as a root. **[4]**

Q2) a) Define an algebraic extension. If E is a finite extension of the field F then prove that E is an algebraic extension of F. Is the converse of the above result true? Justify. **[8]**

b) Find the inverse of $1 - \sqrt[3]{2} + \sqrt[3]{4}$ in the field $Q\left(\sqrt[3]{2}\right)$. **[8]**

Q3) a) Define an algebraically closed field. Prove that the field K is algebraically closed if and only if every irreducible polynomial in K[x] is of degree 1. **[8]**

b) Prove that an algebraically closed field cannot be finite. **[4]**

c) If E is an extension of the field F and [E:F] is prime then prove that there are no fields properly between E and F. **[4]**

Q4) a) If p is a prime then prove that $f(x) = x^p - 1$ in Q[x] has splitting field $Q(\alpha)$ where $\alpha \neq 1$ and $\alpha^p = 1$. Also prove that $[Q(\alpha) : Q] = p - 1$. **[8]**

b) Find the conditions on a and b such that the splitting field of $x^3 + ax + b \in Q[x]$ has **[8]**

i) Degree of extension 3 over Q;

ii) Degree of extension 6 over Q.

P.T.O.

Q5) a) If F is finite field then prove that there exist an irreducible polynomial of any given degree n over F. **[8]**

b) Investigate whether a finite field with following number of elements exist and if so construct such a field. **[8]**

i) 625

ii) 70.

Q6) a) Define **[8]**

i) A separable extension.

ii) Perfect field.

Prove that an algebraic extension of a perfect field is separable.

b) If K is a field of characteristic $p \neq 0$ and if $K^p = K$ then prove that K is a perfect field. **[4]**

c) Prove that any extension E of a field F, such that [E:F]=2 is a normal extension. **[4]**

Q7) a) If E is a finite separable extension of a field F and if $[E:F] = |G(E/F)|$ then prove that F is the fixed field of G(E/F). **[6]**

b) Show that the group $G\left(Q(\sqrt[3]{2})/Q\right)$ is trivial group. **[5]**

c) Show that a finite field is a Galois extension of any of its subfields. **[5]**

Q8) a) If E is a Galois extension of F and K is any subfield of E containing F then prove that **[6]**

$[E:K] = |G(E/K)|$ and

[K:F] = index of G(E/K) in G(E/F).

b) Find the splitting field of $x^4 + 1 \in Q[x]$ and show that it is a Galois extension of Q. **[4]**

c) Show that the polynomial $x^7 - 10x^5 + 15x + 5$ is not solvable by radicals over Q. **[6]**

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Total No. of Questions : 8]

SEAT No. :

P2066

[Total No. of Pages :3

[4821]-42
M.A./M.Sc.
MATHEMATICS
MT-802: Combinatorics
(2008 Pattern) (Semester-IV)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right side indicate full marks.*

Q1) a) How many ways can a committee be formed from four men and six women with **[6]**

i) At least two men and at least twice as many women as men.

ii) Four members, at least two of whom are women,and Mr. and Mrs. Baggins cannot both be chosen?

b) How many ways are there to place an order for 12 chocolate sundaes if there are 5 types of sundaes, and at most 4 sundaes of one type are allowed? **[6]**

c) Find the rook polynomial for a full $n \times n$ board. **[4]**

Q2) a) How many integers between 1000 and 10,000 are there with **[6]**

i) Repetition of digits allowed but with no 2 or 4?

ii) Distinct digits and at least one of 2 and 4 must appear?

b) If 10 steaks and 15 lobsters are distributed among four people, how many ways are there to give each person at most 5 steaks and at most 5 lobsters? **[6]**

c) Verify the identity by a committee selection model. **[4]**

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots\dots + \binom{n}{n}^2 = \binom{2n}{n}.$$

P.T.O.

Q3) a) How many ways are there to form a committee of 10 mathematical scientists from a group of 15 mathematicians, 12 statisticians and 10 operations researchers with at least one person of each different profession on the committee? **[6]**

b) How many numbers greater than 3,000,000 can be formed by arrangements of 1, 2, 2, 4, 6, 6, 6? **[6]**

c) Show that any subset of eight distinct integers between 1 and 14 contains a pair of integers *k,l* such that *k* divides *l*. **[4]**

Q4) a) How many ways are there to distribute 18 chocolate doughnuts, 12 cinnamon doughnuts and 14 powdered sugar doughnuts among four school principals if each principal demands at least 2 doughnuts of each kind? **[6]**

b) Solve the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2} + 3, \quad a_0 = a_1 = 1.$ **[6]**

c) How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 \le 15$ with $x_i \ge -10$ for i=1, 2, 3, 4? **[4]**

Q5) a) Find ordinary generating function with $a_r = r(r+2)$. **[6]**

b) A school has 200 students with 80 students taking each of the three subjects: trigonometry, probability and basket-weaving. There are 30 students taking any given pair of these subjects, and 15 students taking all three subjects. **[6]**

i) How many students are taking none of these three subjects?

ii) How many students are taking only probability?

c) How many arrangements of letters in REPETITION are there with the first E occuring before the first T? **[4]**

Q6) a) Using generating functions, solve the recurrence relation, **[6]**

$a_n = 2a_{n-1} + 2^n$ with $a_0 = 1$.

b) How many ways are there to distribute 15 identical objects into four different boxes, if the number of objects in box 4 must be a multiple of 3? **[6]**

c) Find a generating function for the number of integers between 0 and 999, 999 whose sum of digits is r. **[4]**

Q7) a) How many permutations of the 26 letters are there that contain none of the sequences MATH, RUNS, FROM, or JOE? **[6]**

b) How many r-digit quaternary sequences are there in which the total number of 0's and 1's is even? **[6]**

c) Show that any subset of n+1 distinct integers between 2 and 2n $(n \geq 2)$ always contains a pair of integers with no common divisor. **[4]**

Q8) a) Find and solve a recurrence relation for the number of ways to arrange flags on an n-foot pole using three types of flags; red flags 2 feet high, yellow flags 1 foot high and blue flags 1 foot high. **[8]**

b) Five officials O_1, O_2, O_3, O_4, O_5 are to be assigned five different city cars: an Escort, a Lexus, a Nissan, a Taurus and a Volvo. If O_1 will not drive an Escort or Volvo; O_2 will not drive Lexus or Nissan; O_3 will not dirve Nissan; O_4 will not drive Escort or Volvo; O_5 will not drive Nissan. How many ways are there to assign the officials to different cars? **[8]**

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Total No. of Questions : 8]

SEAT No. :

P2067

[Total No. of Pages :2

[4821]-43
M.A./M.Sc.
MATHEMATICS
MT:803- Differentiable Manifolds
(2008 Pattern) (Semester-IV)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Define the terms: **[4]**

i) Volume of a parametrized manifold.

ii) Centroid of a parametrized manifold.

b) Let M be a manifold in $\mathbb{R}^n$ and $\alpha : \text{U} \to \text{V}$ be a coordinate patch on M. If U_0 is a subset of U, then prove that the restriction of α to U_0 is also a coordinate patch in M. **[7]**

c) Give an example of a 1-manifold in $\mathbb{R}^2$ which can be covered by a single coordinate patch. **[5]**

Q2) a) Let M be an oriented k-manifold with non-empty boundary, then prove that ∂M is orientable. **[7]**

b) Show that n-ball $\text{B}^n(a)$ is an n-manifold in $\mathbb{R}^n$.Find its boundary. **[6]**

c) Give an example of a 2-manifold in $\mathbb{R}^3$. **[3]**

Q3) a) What is the dimension of $\text{A}^k(\text{V})$, the space of alternating k-tensors on an n-dimensional space. Justify. **[7]**

b) Let f and g be tensors on $\mathbb{R}^4$ given by $f(x,y,z) = \text{x}_1\text{y}_2\text{z}_2 - x_2y_3z_1$ and $g = \phi_{2,1} - 5\phi_{3,1}$. Express $f \otimes g$ as a linear combination of elementary 5-tensors. **[5]**

c) If $w = xyzdx + (x+y+z)dy + (xy+xz+yz)dz$, then find dw. **[4]**

P.T.O.

Q4) a) Let $w = x^2 z dx + xy dy + z^2 y dz$ and $\alpha(u,v) = (u-v, u+v, u^2)$. Find $\alpha^*(dw)$. **[7]**

b) State Green's theorem. **[4]**

c) Let M be a compact k-manifold in $\mathbb{R}^n$. Define volume of M. **[5]**

Q5) a) For any k-form w, show that $d(dw)=0$. **[7]**

b) Give an example of an alternating tensor. **[4]**

c) State Stoke's theorem. **[5]**

Q6) a) If *f* and *g* are alternating tensors of order *k* and *l* respectively, then prove that $g \wedge f = (-1)^{kl} \; f \wedge g$. **[6]**

b) If $w = xz dx + y^2 dy + xe^y dz$ and $\eta = y \sin x dx + z dy + yz dz$, then find $w \wedge \eta$. **[6]**

c) Give an example of a closed form. **[4]**

Q7) a) Let $\text{A} = (0,1)^2$. Let $\alpha : \text{A} \to \mathbb{R}^3$ be given by the equation $\alpha(u,v) = (u, v, u^2 + v^2 + 1)$. Let Y_α be the image set of α. Evaluate the integral over Y_α, the following two-form: $y\, dy \wedge dz + xz dx \wedge dz$. **[8]**

b) With usual notation, prove that $\alpha^*(dw) = d(\alpha^* w)$. **[8]**

Q8) a) If w and η are k and l forms respectively, then prove that $d(w \wedge \eta) = dw \wedge \eta + (-1)^k w \wedge d\eta$. **[8]**

b) Give an example of a c^∞-function $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ such that α^{-1} is continuous, but $\text{D}\alpha$ is not of rank 2. **[4]**

c) Let $f(x, y, z) = x \sin y + y \cos z + ze^x$. Find *df*. **[4]**

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Total No. of Questions : 8]

SEAT No. :

P2068

[Total No. of Pages :2

[4821]-44
M.A./M.Sc.
MATHEMATICS
MT:804 - Algebraic Topology
(2008 Pattern) (Semester-IV)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right side indicate full marks.***

Q1) a) If $A \subseteq X$ and $f_0, f_1 : X \to Y$ are continuous, when are f_0 and f_1 said to be homotopic relative to A. **[4]**

b) Prove that the relation of being homotopic relative to A is an equivalence relation. **[6]**

c) Let $f, g : X \to S^n$ be continuous maps such that $f(x) \neq -g(x)$ for all $x \in X$. Prove that *f* is homotopic to *g*. **[6]**

Q2) a) When do two spaces X and Y have the same homotopy type? **[4]**

b) If X and Y are homomorphic, prove that they have the same homotopy type. Is the converse true? Why? **[6]**

c) Prove that S^n is a strong deformation retract of $\mathbb{R}^{n+1} \setminus 0$. **[6]**

Q3) a) Prove that the continuous image of a path connected space is path connected. **[4]**

b) Is $S^n, n \geq 1$, path connected? Why? **[6]**

c) If *f* is *a* path in X, prove that $f_* \bar{f}$ is homotopic to a null path. **[6]**

Q4) a) Define the fundamental group $\pi_1(X, x_0)$. **[4]**

b) If $f : X \to Y$ is continuous , define the in duced map $f^* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$. Prove that this map is a homomorphism. **[6]**

c) Find the fundamental group of the projective plane. **[6]**

P.T.O.

Q5) a) When is $P:\tilde{X} \to X$ said to be a covering map? **[4]**

b) Prove that $P:S^1 \to S^1$ geven by $P(z)=z^2$ is a covering map. **[6]**

c) Find a map $f:[0,2\pi]\times[0,2\pi] \to \mathbb{R}^3$ whose image is a focus. What is the fundamental group of a torus? **[6]**

Q6) a) Let a group G act on a space X. Prove that any two orbit of this action are either disjoint or equal. **[4]**

b) Give an action of $\mathbb{Z}$ on $\mathbb{R}$ and prove that it is an action. What are the orbits? **[6]**

c) Let $P:\tilde{X} \to X$ be a covering map, and $f_1, f_2 : Y \to \tilde{X}$ be two liftings of $f:Y \to X$. Suppose Y is connected, and $\exists\ y_0 \in Y$ with $f_1(y_0) = f_2(y_0)$. Prove that $f_1 = f_2$. **[6]**

Q7) a) When is a map $P:E \to B$ called a fibration? **[4]**

b) Give an example of a fibration. **[6]**

c) Prove that a fibration has unique path lifting if and only if every fibre has no non null path. **[6]**

Q8) a) What is a geometric p-simplex? **[4]**

b) Let $e_1=(1, 0, 0)$, $e_2=(0, 1,0)$, $e_3=(0,0,1)$, $e_0=(0, 0, 0)$. Draw the 0,1,2 and 3 faces of $s_3=\{e_0, e_1, e_2, e_3\}$. **[6]**

c) Is the surface of the unit sphere in $\mathbb{R}^3$ a triangulable space? Why? **[6]**

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Total No. of Questions : 8]

SEAT No. :

P2069

[Total No. of Pages :2

[4821]-45

M.A./M.Sc.

MATHEMATICS

MT-805 : Lattice Theory

(2008 Pattern) (Semester-IV)

Time : 3 Hours] *[Max. Marks : 80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Let $\mathbb{N}_0$ be the set of all non-negative integers. Define $m \le n$ if and only if there exists $k \in \mathbb{N}_0$ such that $n = km$. Prove that $\mathbb{N}_0$ is a lattice under this relation. **[6]**

b) Define a congruence relation on a lattice *L* and prove that the set of all congruence relations on *L* forms a lattice. **[6]**

c) Prove that every homomorphism is an isotone map but not conversely. **[4]**

Q2) a) Prove that a lattice *L* is distributive if and only if for any two ideals *I*,*J* of *L*, $I \vee J = \{i \vee j \mid i \in I, j \in J\}$. **[6]**

b) Prove that in a distributive lattice *L*, if the ideals $I \vee J$ and $I \wedge J$ are principal then so are *I* and *J*. **[6]**

c) Let *L* be a finite distributive lattice. Prove that *L* is pseudocomplemented. Is finiteness necessary to prove the assertion. Justify your answer. **[4]**

Q3) a) Show that the following inequalities hold in any lattice. **[6]**

i) $(x \wedge y) \vee (x \wedge z) \le x \wedge (y \vee z)$;

ii) $(x \wedge y) \vee (x \wedge z) \le x \wedge (y \vee (x \wedge z))$

b) Prove that in a distributive lattice *L*, the element $a \ne 0$ is join-irreducible if and only if *L*-[*a*) is a prime ideal. **[6]**

c) Let *L* be a distributive lattice. Show that *Id*(*L*), the ideal lattice of *L*, is distributive. **[4]**

P.T.O.

Q4) a) Prove that every maximal chain C of the finite distributive lattice L is of length $|J(L)|$. **[6]**

b) Prove that every prime ideal is a meet-irreducible element of the ideal lattice but not conversely. **[6]**

c) Prove that every distributive lattice is modular but not conversely. **[4]**

Q5) a) Let L be a pseudocomplemented lattice. Prove that $S(L) = \{a^* \mid a \in L\}$ is a bounded lattice. **[8]**

b) Prove that a lattice is modular if and only if it does not contain a pentagon (N_5) as a sublattice. **[8]**

Q6) a) Prove that a lattice is distributive if and only if it is isomorphic to ring of sets. **[8]**

b) Let L be a bounded distributive lattice with $|L| > 1$ and the set $P(L)$ of all prime ideals of L is unordered. Prove that L is complemented. **[8]**

Q7) a) Prove that every prime ideal of a Boolean lattice is maximal and conversely. **[8]**

b) Let L be a distributive, lattice, let I be an ideal, let D be a dual ideal of L, and let $I \cap D = \phi$. Then prove that there exists a prime ideal P of L such that $P \supseteq I$ and $P \cap D = \phi$. **[8]**

Q8) a) State and prove Jordan-Hölder Theorem for semimodular lattices. **[7]**

b) Prove that in a finite distributive lattice, $|J(L)| = |M(L)|$, where $J(L)$ and $M(L)$ are the of join-irreducibles and meet-irreducibles of L. **[5]**

c) Prove that every modular lattice is semimodular but not conversely. **[4]**