Total No. of Questions : 8]

SEAT No. :

P2070

[Total No. of Pages :3

[4821]-101
M.A./M.Sc.
MATHEMATICS
MT-501 : Real Analysis
(2013 Pattern) (Semester-I) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Let $E \subset \mathbb{R}^d$ and $E = \bigcup_{j=1}^{\infty} E_j$ then prove that $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$. **[5]**

b) Define a Lebesgue measurable set E. If E is measurable then show that E^c is also measurable. **[3]**

c) Show that Lebesgue measure is translation invarient. **[2]**

Q2) a) If f and g are measurable functions then show that $f^k (k \geq 1)$ and f.g are also measurable. Also give an example of a measurable function which is not continuous. **[5]**

b) Show that if f is a measurable function then |f| is also measurable. Is converse true? Justify. **[3]**

c) State Egorov theorem. **[2]**

Q3) a) If f is a non-negative, bounded and measurable function defined on a measurable set E and $\int_E f = 0$ then prove that $f(x) = 0$ *a.e.* on E. **[5]**

b) Let $f(x) = \frac{1}{x}, 0 < x \leq 1$

$= 19$ for $x = 0$ **[3]**

then show that f is not Lebesgue integrable on [0,1].

c) Show that a continuons function is Lebesgue measurable function. **[2]**

P.T.O.

Q4) a) State and prove bounded convergence theorem for a sequence of measurable functions which are bounded and supported on a set of finite measure. **[5]**

b) Suppose $\{f_n\}$ is a sequence of measurable functions with $f_n \geq 0$ and $f_n \uparrow f$ then prove $\lim_{n\to\infty} \int f_n = \int f$. **[3]**

c) Show that strict inequality can hold in Fatou's lemma. **[2]**

Q5) a) State and prove Riesz - fischer theorem. **[7]**

b) Which of the following families of functions are dense in $L^1(\mathbb{R}^d)$? **[3]**

i) Simple functions

ii) Step functions

iii) Continuons functions with compact support.

Q6) a) Give an example of a positive function *f* defined on $\mathbb{R}$ such that f is integrable but $\lim_{|x|\to\infty} \sup f(x) = \infty$. **[4]**

b) Show that if *f* and *g* are integrable on $\mathbb{R}^d$ then $f(x-y)\,g(y)$ is integrable on $\mathbb{R}^{2d}$. **[4]**

c) State Fubini's theorem. **[2]**

Q7) a) State Lebesgue differentiation theorem. **[2]**

b) Give an example of a continuous function which is not absolutely continuous. **[4]**

c) Define good kernels. **[4]**

Which of the following are good kernels?

(1) Dirichlet kernels (2) Poisson kernel

Q8) a) Let $a > b > 0$ then prove that

$f(x) = x^a \sin (x^{-b}), 0 < x \le 1$

$f(0) = 0$ is of bounded variation. **[5]**

b) Define Dini numbers $D^+F(x), D_+F(x), D^-F(x), D_-F(x)$ and calculate Dini numbers for F at $x = 0$ where $F(x) = x\sin\left(\frac{1}{x}\right)$ if $x \ne 0 \in \mathbb{R}$

$=0$ if $x = 0$ **[5]**

Total No. of Questions : 8]

SEAT No. :

P2071

[Total No. of Pages :3

[4821]-102

M.A./M.Sc.

MATHEMATICS

MT-502 : Advanced Calculus

(2013 Pattern) (Semester-I) (Credit System)

Time : 3 Hours] ***[Max. Marks : 50***

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) i) Define a directional derivative of a scalar field.

ii) State true or false.

If all the directional derivative of a scalar field exist at a point then f is differentiable at that point. Justify. **[5]**

b) If $\lim_{(x,y)\to(a,b)} f(x, y) = L$ and if the one dimensional limits $\lim_{x\to a} f(x, y)$ and $\lim_{y\to b} f(x, y)$ both exist, prove that $\lim_{x\to a}\left[\lim_{y\to b} f(x, y)\right] = \lim_{y\to b}\left[\lim_{x\to a} f(x, y)\right] = L$ **[3]**

c) State the implicit function theorem. **[2]**

Q2) a) State and prove chain rule for derivatives of vector field. **[5]**

b) Let $\overline{f} : \mathbb{R}^2 \to \mathbb{R}^2$ and $\overline{g} : \mathbb{R}^3 \to \mathbb{R}^2$ be two vector fields defined as follows;

$\overline{f}(x, y) = e^{x+2y}\overline{i} + \sin(y + 2x)\overline{j}$

$\overline{g}(u, v, w) = (u + 2v^2 + 3w^3)\overline{i} + (2v - u^2)\overline{j}$

i) Compute each of the Jacobian matrices D $\overline{f}(x, y)$ and D $\overline{g}(u, v, w)$.

ii) Compute the composition $\overline{h}(u, v, w) = \overline{f}[\overline{g}(u, v, w)]$

iii) Compute the Jacobian matrix D $\overline{h}(1, -1, 1)$. **[3]**

c) Let the two equations $x = e^u \cos v$ $\quad y = e^u \sin v$ defines $u = u(x, y)$ and $v = v(x, y)$. Find explicit formulas for $u(x, y)$ and $v(x, y)$ when $x > 0$. **[2]**

Q3) a) Define line integral of vector field. A force field $\bar{f}$ in 3-space is given by $\bar{f}(x, y, z) = x\bar{i} + y\bar{i} + (xz - y)\bar{k}$. Compute the work done by this force in moving a particle from (0, 0, 0) to (1, 2, 4) along the line segment joining these two points. **[4]**

b) Compute the moment of inertia I_z about that z-co-ordinate axis of the spring coil having the shape of the helix whose vector equation is $\bar{\alpha}(t) - a\cos t\,\bar{i} + a\sin t\,\bar{j} + bt\,\bar{k}$ if the density at (*x, y, z*) is $x^2 + y^2 + z^2$. **[3]**

c) If the vector fields $\bar{f}, \bar{g} : \mathbb{R}^n \to \mathbb{R}^m$ are continuous at $\bar{a}$, then prove that their dot product is also continuous at $\bar{a}$. **[3]**

Q4) a) Let φ be a differentiable scalar field with a continuous gradient $\nabla\varphi$ on an open connected set S in $\mathbb{R}^n$ prove that for any two points $\bar{a}$ and $\bar{b}$ joined by a piecewise smooth path $\bar{\alpha}$ in S we have

$$\int_a^b \nabla\varphi . d\bar{\alpha} = \varphi(\bar{b}) - \varphi(\bar{a}).$$ **[5]**

b) Calculate the line integral of the vector field $\bar{f}(x, y, z) = x\bar{i} + y\bar{j} + (xz - y)\bar{k}$ along the path described by $\bar{\alpha}(t) = t^2\bar{i} + 2t\bar{j} + 4t^3\bar{k} \quad 0 \le t \le 1$. **[3]**

c) Determine whether or not the vector field $\bar{f}(x, y) = 3x^2 y\,\bar{i} + x^3 y\,\bar{j}$ is a gradient on any open subset of $\mathbb{R}^2$. **[2]**

Q5) a) Define double integral of a step function. Explain how to use this to define the double integral of a function which is defined and bounded on rectangle. **[5]**

b) Evaluale $\iint_\theta xy(x + y)\,dx\,dy$ where $\theta = [0,1] \times [0,1]$. **[3]**

c) Transform the integral $\int_0^1 \left[\int_0^1 f(xy)dx\right] dy$ to one or more iterated integrals in polar co-ordinate. **[2]**

Q6) a) Define the fundamental vector product, show that the fundamental vector product is normal to the surface. **[5]**

b) Let $\bar{r}$ and $\bar{R}$ be smoothly equivalent functions related by the equation $\bar{R}(s,t)=\bar{r}(\bar{G}(s,t))$ where $\bar{G}(s,t)=U(s,t)\bar{i}+V(s,t)\bar{j}$ is a one-one continuously differentiable mapping of a region B in the st-plane onto a region A in the uv-plane. Prove that $\frac{\partial \bar{R}}{\partial s}\times\frac{\partial \bar{R}}{\partial t}=\left(\frac{\partial \bar{r}}{\partial u}\times\frac{\partial \bar{r}}{\partial v}\right)\frac{\partial(u,v)}{\partial(s,t)}$ where the partial derivatives are evaluated at the point $(u(s,t),v(s,t))$. **[3]**

c) Compute the fundamental vector product $\bar{r}(u,v)=au\cos v\bar{i}+bu\sin v\bar{j}+u^2\bar{k}$. **[2]**

Q7) a) Let P be a scalar field which is continuously differentiable on an open set S in the xy-plane. Let C be a piecewise smooth Jordan curve, and R denote the union of C and its interior, suppose R is of special type R=$\{(x,y)\,|\,a\le x\le b \text{ and } f(x)\le y\le g(x)\}$ where f and g are continuous on [a,b] with $f\le g$ and assume R is subset of S. Prove that $\iint_R \frac{\partial P}{\partial y}dx\,dy=\oint_C P\,dx$ where the line integral is taken around C in the counter clockwise direction. **[5]**

b) Explain the change of variables in a double integral by polar co-ordinates. Evaluate the integral using polar co-ordinate $\iint_s \sqrt{a^2-x^2-y^2}\,dx\,dy$ where the region S is the first quadrant of the circular disk $x^2+y^2\le a^2$. **[5]**

Q8) a) State the Stokes theorem and show that the surface integral which appears in Stoke's theorem in term of the curl of a vector field. **[5]**

b) Determine the Jacobian matrix and compute the curl and divergence of $\bar{F}$ where $\bar{F}(x,y,z)=xy^2z^2\bar{i}+z^2\sin y\bar{j}+x^2e^y\bar{k}$. **[5]**

Total No. of Questions : 8]

SEAT No. :

P2072

[Total No. of Pages :2

[4821]-103
M.A./M.Sc.
MATHEMATICS
MT-503: Group Theory
(2013 Pattern) (Semester-I) (Credit System)

Time : 3 Hours] ***[Max. Marks : 50***

Instructions to the candidates:

1) ***Figures to right indicate full marks.***
2) ***Attempt any five questions.***

Q1) a) Show that U(n) is a group under multiplication modulo n. **[4]**

b) Show that for each element a in G there exists unique element $b \in G$ such that $a * b = b * a = e$ where e is identity element in G. **[3]**

c) Find the order of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ in SL $(2, \mathbb{R})$. **[3]**

Q2) a) State and prove fundamental theorem for the cyclic groups. **[5]**

b) Show that the image of a cyclic group under isomorphism is cyclic. **[3]**

c) Show that every cyclic group is abelian. **[2]**

Q3) a) Determine the subgroup lattice for $\mathbb{Z}_{12}$. **[4]**

b) Show that the center of group is a subgroup of the group. **[3]**

c) Show that a group of order p^2, p prime is abelian. **[3]**

Q4) a) Verify Cayley's theorem for U (10). **[5]**

b) Show that group of even permutations in S_n form a subgroup of S_n. **[5]**

Q5) a) State and prove orbit stabilizer theorem for the finite group of permutations. **[5]**

b) Show that rotations of a cube is isomorphic to S_4. **[5]**

Q6) a) If G and H are finite cyclic groups of prime orders p and q respectively. (p,q)=1 show that $G \oplus H$ is cyclic. **[5]**

b) Find $U_7(105)$. **[3]**

c) List all cyclic subgroups of D_4. **[2]**

Q7) a) Show that G/Z (G) is isomorphic to $I_{nn}(G)$. **[3]**

b) State Sylow's three theorem's for the finite groups along with statement of Cauchy's theorem. **[4]**

c) Show that the only group of order 255 is $\mathbb{Z}_{255}$. **[3]**

Q8) a) State and prove first isomorphism theorem for groups. **[5]**

b) If $\phi: G \to G'$ is a group homomorphism with $\phi(g) = g'$ show that $\phi^{-1}(g') = g \ker \phi$. **[5]**

Total No. of Questions : 8]

SEAT No. :

P2073

[Total No. of Pages :3

[4821]-104
M.A./M.Sc.
MATHEMATICS
MT-504: Numerical Analysis
(2013 Pattern) (Semester-I) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***
3) ***Use of non-programmable, Scientific calculator is allowed.***

Q1) a) Prove that the order of convergence of the secant method is approximately 1.618 ($\alpha = 1.618$) and asymptotic error constant $\lambda \approx c^{1/\alpha} = \left(\frac{f''(p)}{2f'(p)}\right)^{\alpha-1}$. **[5]**

b) Show that when Newton's method is applied to the equation $\frac{1}{x} - a = 0$, the resulting iteration function is $g(x) = x(2-ax)$. **[3]**

c) Determine the rate of convergence of the function $f(x) = \frac{\cos x - 1 + \frac{1}{2}x^2}{x^4}$. **[2]**

Q2) a) Show that the function $g(x) = e^{-x}$ has a unique fixed point near x=0.6 by using fixed point Iteration method and starting value $p_0 = 0$. (Do at least 5 Iterations) **[5]**

b) Verify that the equation $x^4 - 18x^2 + 45 = 0$ has a root on the interval (1, 2), perform five iterations of the secant method, using p_0=1 and p_2 = 2. **[3]**

c) Define the terms:

i) Rate of convergence of sequence

ii) Asymptotic error constant. **[2]**

P.T.O.

Q3) a) Solve the following system of equation by using Gaussian elimination with partial pivoting. **[5]**

$$x+y+z=0$$
$$3u+3y-4z=7$$
$$u+x+y+2z=6$$
$$2u+3x+y+3z=6$$

b) Determine the crout decomposition of the given matrix and then solve the system Ax = b for the right hand side vector. **[3]**

$$[A]=\begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix}, \ b=\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$$

c) Compute the condition number K_∞ for the matrix **[2]**

$$A=\begin{bmatrix} 1 & -2 \\ -0.99 & 1.99 \end{bmatrix}$$

Q4) a) Solve the following system of linear equations by Gauss-Seidel method, start with $x^{(0)}=[0\ 0\ 0]^T$. (Perform 3 iterations) **[5]**

$$4x_1 - x_2 = 2$$
$$-x_1 + 4x_2 - x_3 = 4$$
$$-x_2 + 4x_3 = 10$$

b) Solve the following system of linear equations by SOR method, start with $x^{(0)}=[0\ 0\ 0]^T$ and w=0.9. (Perform 2-iterations). **[3]**

$$5x_1 + x_2 + 2x_3 = 10$$
$$-3x_1 + 9x_2 + 4x_3 = -14$$
$$x_1 + 2x_2 - 7x_3 = -33$$

c) Show that the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ has no LU decomposition. **[2]**

Q5) a) Solve the following non-linear system of equation by Newtons method (starting vector $x^{(0)}=[1\ 1\ 1]^T$).

$$x_1^3 - 2x_2 - 2 = 0$$
$$x_1^3 - 5x_3^2 + 7 = 0$$
$$x_2 x_3^2 - 1 = 0$$

(Perform 2-iterations) **[5]**

b) Construct the Householder matrix H for $W = \left[\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}\ \frac{1}{2}\right]^T$. **[3]**

c) For the non-linear system

$$x_1 + x_2 - x_1^3 = 0$$
$$x_1 + x_2 + x_2^3 = 0$$

Compute the Jacobian of F. **[2]**

Q6) a) Derive the closed Newton-cotes formula with n = 3; **[5]**

$$\int_a^b f(x)dx = \frac{b-a}{8}\left[f(a)+3f(a+\Delta x)+3f(a+2\Delta x)+f(b)\right]$$

b) Derive the following forward difference approximation for the second derivative $f''(x_0) \approx \frac{f(x_0)-2f(x_0+h)+f(x_0+2h)}{h^2}$ **[3]**

c) If $f(x)=\ln(x)$, find $f'(2)$ for h = 0.1, 0.01. **[2]**

Q7) a) Determine the degree of precision of the Simpson's $\frac{3}{8}^{th}$ rule. **[5]**

b) Use Householder's method to reduce the following symmetric matrix to tridiagonal form

$$[A] = \begin{bmatrix} -1 & -2 & 1 & 2 \\ -2 & 3 & 0 & -2 \\ 1 & 0 & 2 & 1 \\ 2 & -2 & 1 & 4 \end{bmatrix}$$ **[5]**

Q8) a) Apply Euler's method to approximate the solution of the initial value problem $\frac{dx}{dt} = tx^3 - x$, $(0 \le t \le 1)$, $x(0) = 1$ using 4 steps. **[5]**

b) Use the Runge-Kutta method of order 4, to solve initial value problem

$$\frac{dx}{dt} = 1 + \frac{x}{t},\ 1 \le t \le 6,\ x(1) = 1$$ **[5]**

Total No. of Questions : 8]

SEAT No. :

P2074

[Total No. of Pages :3

[4821]-105

M.A./M.Sc.

MATHEMATICS

MT-505: Ordinary Differential Equations

(2013 Pattern) (Semester-I) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) If $y_1(x)$ is one solution of the differential equation $y'' + P(x)y' + Q(x)y = 0$, then find the other solution. **[5]**

b) Find the general solution of $y'' + 4y = 3\sin x$ by using method of undetermined coefficients. **[3]**

c) Show that $y_1 = e^x$ and $y_2 = e^{-x}$ are linearly independent solutions of $y'' - y = 0$ on any interval. **[2]**

Q2) a) Discuss the method of variation of parameters to find the solution of second order differential equation with constant coefficients. **[5]**

b) Change the independent variable x by $x = e^z$ and solve the differential equation $2x^2y'' + 10xy' + 8y = 0$. **[3]**

c) Verify that $y_1 = x^2$ is one solution of $x^2y'' + xy' - 4y = 0$ and find y_2 and the general solution. **[2]**

Q3) a) State and prove sturm comparison theorem. **[5]**

b) Let u (x) be any nontrivial solution of $u'' + q(x)u = 0$, where q (x) > 0 for all $x > 0$. If $\int_1^\infty q(x)dx = \infty$, then prove that u($x$) has infinitely many zeros on the positive x-axis. **[3]**

c) Show that $y = c_1e^x + c_2e^{2x}$ is general solution of $y'' - 3y' + 2y = 0$ on any interval. **[2]**

P.T.O.

Q4) a) Find the general solution of differential equation $(1+x^2)y''+2xy'-2y=0$ in terms of power series x. **[5]**

b) Find the indicial equation and its roots of the differential equation $2xy''+(3-x)y'-y=0$. **[3]**

c) Locate and classify the singular points on the x-axis of $x^3(x-1)y''-2(x-1)y'+3xy=0$. **[2]**

Q5) a) Find two independent Frobenius series solution of the differential equation $4xy''+2y'+y=0$. **[5]**

b) For the following system

$$\frac{dx}{dt}=-y$$

$$\frac{dy}{dt}=x$$

i) Find the critical points.

ii) Find the differential equation of path.

iii) Solve the equation to find the path. **[3]**

c) Show that the series $y=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+----$ is solution of differential equation $y''+y=0$. **[2]**

Q6) a) Find the general solution of the system. **[5]**

$$\frac{dx}{dt}=x+y$$

$$\frac{dy}{dt}=4x-2y$$

b) Determine the nature of the point at $x=\infty$ for the differential equation $x^2y''+xy'+(x^2-4)y=0$. **[3]**

c) State Picard's existence and uniqueness theorem. **[2]**

Q7) a) If m_1 and m_2 are roots of the auxiliary equation of the system

$$\frac{dy}{dt} = a_2 x + b_2 y$$

$$\frac{dy}{dt} = a_2 x + b_2 y$$

Which are real, distinct and of opposite sign, then prove that critical point (0, 0) is saddle point. **[5]**

b) Find the general solution near $x = 0$ of the hypergeometric equation $x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$ where a, b, c are constants. **[5]**

Q8) a) If f (x, y) be a continuous function that satisfies a Lipschitz condition $|f(x, y_1) - f(x, y_2)| \le k|y_1 - y_2|$ on a strip defined by $a \le x \le b$ and $-\infty < y < \infty$. If (x_0, y_0) is any point of the strip, then prove that the intial value problem $y' = f(x, y)$, $y(x_0) = y_0$ has one and only one solution $y = f(x)$ on the interval $a \le x \le b$. **[5]**

b) Solve the following initial value problem. **[5]**

$$\frac{dy}{dx} = z,\ y(0) = 1$$

$$\frac{dz}{dx} = -y,\ z(0) = 0\,.$$

Total No. of Questions : 8]

SEAT No. :

P2075

[Total No. of Pages :3

[4821]-201
M.A./M.Sc.
MATHEMATICS
MT-601: Complex Analysis
(2013 Pattern) (Semester-II) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Suppose $f = u + iv$ is a complex-valued function defined on an open set Ω . If u and v are continuously differentiable and satisfy the Cauchy-Riemann equations on Ω, then prove that f is holomorphic. **[4]**

b) Show that in polar coordinates, the Cauchy- Riemann equations take the form $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$. Use these equations to show that the logarithmic function defined by log z = log r + i θ where $z = re^{i\theta}$ with $-\pi < \theta < \pi$ is holomorphic in the region r >0 and $-\pi < \theta < \pi$. **[3]**

c) If a continuous function f has a primitive f in Ω, and γ is a curve in Ω that begins at w_1 and ends at w_2, then prove that $\int_\gamma f(z)dz = F(w_2) - F(w_1)$. **[3]**

Q2) a) Given a power series $\sum_{n=0}^{\infty} a_n z^n$, prove that there exists $0 \le R \le \infty$ such that

i) If |z| < R the series converges absolutely.

ii) If |z| > R the series diverges. **[5]**

b) Let γ be a smooth curve in $\mathbb{C}$ parametrized by $z(t) = [a,b] \to \mathbb{C}$. Let γ^- denote the curve with the same image as γ but the reverse orientation. Prove that for any continuons function f on γ $\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz$. **[3]**

c) Determine the radius of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$ when

i) $a_n = (\log n)^2$ ii) $a_n = n!$ **[2]**

P.T.O.

Q3) a) If Ω is an open set in $\mathbb{C}$, and T $\subset \Omega$ a triangle whose interior is also contained in Ω, then prove that $\int_T f(z)dz = 0$ whenever f is holomorphic in Ω. **[5]**

b) Prove that $\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$. **[5]**

Q4) a) If f is entire and bounded, then prove that f is constant. Also prove that every non-constant polynomial with complex coefficients has a root in $\mathbb{C}$. **[4]**

b) If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then prove that f is holomorphic in Ω. **[4]**

c) Evalute the integral $\int_{|z|=1} \frac{z^2}{2z+1}dz$ **[2]**

Q5) a) Let F(z,s) be defined for $(z,s) \in \Omega \times [0,1]$ where Ω is an open set in $\mathbb{C}$. Suppose that F satisfies the following properties.

i) F (z, s) is holomorphic in z for each s.

ii) F is continuous on $\Omega \times [0,1]$.

Then prove that the function f defined on Ω by $f(z) = \int_0^1 F(z,s)ds$ is holomorphic. **[5]**

b) Suppose f is holomorphic in an open set Ω, and $K \subset \Omega$ is compact. Then prove that there exists finitely many segments $\gamma_1, \gamma_2, \ldots\ldots, \gamma_N$ in $\Omega - K$ such that $f(z) = \sum_{n=1}^{N} \frac{1}{2\pi i} \int_{\gamma_n} \frac{f(\xi)}{\xi - z} d\xi$ for all $z \in K$. **[5]**

Q6) a) Suppose that *f* is holomorphic in a connected open set Ω, has a zero at a point $z_0 \in \Omega$, and does not vanish identically in Ω. Then prove that there exists a neighbourhood $U \subset \Omega$ of z_0, a non-vanishing holomorphic function g on U, and a unique positive integer n such that $f(z) = (z - z_0)^n g(z)$ for all $z \in U$. **[3]**

b) Suppose that f is holomorphic in an open set containing a circle C and its interior, except for a pole at z_0 inside C. Then prove that $\int_C f(z)dz = 2\pi i \operatorname{res}_{z_0} f$. **[3]**

c) Prove that $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$. **[4]**

Q7) a) Suppose that f is holomorphic in an open set Ω except possibly at a point z_0 in Ω. If f is bounded on Ω-$\{z_0\}$, then prove that z_0 is a removable singularity. **[4]**

b) If f is holomorphic and non-constant in a region Ω, then prove that f is open. **[3]**

c) If f is a non-constant holomorphic function in a region Ω, then prove that f cannot attain a maximum in Ω. **[3]**

Q8) a) Prove that the meromorphic functions in the extended complex plane are the rational functions. **[5]**

b) Prove that any holomorphic function in a simphy connected domain has a primitive. **[3]**

c) If f is holomorphic in the simply connected region Ω, then prove that

$\int_\gamma f(z)dz = 0$ for any closed curve γ in Ω. **[2]**

Total No. of Questions : 8]

SEAT No. :

P2076

[Total No. of Pages :2

[4821]-202

M.A.\M.Sc.

MATHEMATICS

MT-602: General Topology

(2013 Pattern) (Semester-II) (Credit System)

Time : 3 Hours] ***[Max. Marks : 50***

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Let x be a set, let τ_c be the collection of all subsets U of x such that X-U is either countable or all of x. Prove that τ_c is a topology on x. **[5]**

b) Let x be a set; let $\mathcal{B}$ be a basis for a topology τ on x. Then, prove that τ equals the collection of all unions of members of $\mathcal{B}$. **[3]**

c) If x ={a, b, c}, let $\tau_1 = \{\phi, x, \{a\}, \{a,b\}\}$ and $\tau_2 = \{\phi, x, \{a\}, \{b,c\}\}$ find the smallest topology containing τ_1 and τ_2 and the largest topology contained in τ_1 and τ_2. **[2]**

Q2) a) Prove that, the collection $S = \{\pi_1^{-1}(\text{U}) \mid \text{U is open in } x\}$U $\{\pi_2^{-1}(\text{V}) \mid \text{V is open in } y\}$is a subbasis for the product topology on X ×Y, where π_1 and π_2 are projections of X × Y on X and Y respectively. **[5]**

b) If X is a Hausdorf f space then show that the diagonal $\Delta = \{x \times x \mid x \in X\}$ is closed in X × X. **[3]**

c) Let y be the subset [0, 1] U {2} of $\mathbb{R}$. Show that the set {2} is open in the subspace topology on Y but is not open in the order topology on y,**[2]**

P.T.O.

Q3) a) If X, Y are topological spaces and $f: X \to Y$ then prove that the following are equivalent. **[5]**

i) f is continuous

ii) $f(\bar{A}) \subset \overline{f(A)}$ for every $A \subset X$

iii) For every closed set B in Y, the set $f^{-1}(B)$ is closed in X.

b) Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection map. Show that π_1 is a continuous, open map but not a closed map. **[3]**

c) Find the boundary and interior of $C \subset \mathbb{R}^2$ where $C = A \cup B$, $A = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}, B = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y \neq 0\}$. **[2]**

Q4) a) Prove that union of a collection of connected sets that have a point in common is connected. **[5]**

b) Show that the space I × I in the dictionary order topology is a linear continuum. **[3]**

c) Show that $\bar{S}$ is connected set in $\mathbb{R}^2$ where $S = \left\{ x \times \sin\left(\frac{1}{x}\right) \mid 0 < x \leq 1 \right\}$. **[2]**

Q5) a) Prove that a topological space x is locally connected if and only if for every open set U of X, each component of U is open in X. **[5]**

b) Let $f: X \to Y$ be a bijective continuous map. If X is compact and Y is Hausdorff, then prove that f is a homeomorphism. **[3]**

c) Determine the components of discrecte topological space. **[2]**

Q6) a) State and prove the tube lemma. **[5]**

b) Prove that compactness implies limit point compactness. Is the converse true? Justify. **[3]**

c) Give an example to show that a locally compact space need not be compact. **[2]**

Q7) a) Prove that a subspace of a first countable space is first countable and a countable product of first countable spaces is first countable. **[5]**

b) Prove that every compact Hausdorff space is normal. **[3]**

c) Prove that product of regular spaces is regular. **[2]**

Q8) a) State and prove Tychonoff theorem. **[8]**

b) State Tietze extension theorem. **[2]**

Total No. of Questions : 8]

SEAT No. :

P2077

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[4821]-203
M.A.\M.Sc.
MATHEMATICS
MT-603: Ring Theory
(2013 Pattern) (Semester-II) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) If R is a commutative ring with unity 1 then prove that $A \in M_n(R)$ is a unit if and only if its determinant, det (A), is a unit in R. **[4]**

b) Show that the ring $R = Z\left[\sqrt{-2}\right]$ has no units other than $\pm$ 1. Also show that the ring $S = Z\left[\sqrt{2}\right]$ has infinitely many units. **[4]**

c) Prove or disprove

For any prime p and a positive integer n, every zero-divisor in Z_{p^n} is nilpotent. **[2]**

Q2) a) If for $n \geq 2$, the ring Z_n has no non-trivial nilpotent elements, then prove that n is square free. What can you say about the converse? Justify. **[5]**

b) Show that the principal ideal (1 + i) in z [i] is maximal. **[3]**

c) Prove or disprove

In the ring R=C $([0,1], \mathbb{R})$, the ideal (0) is prime. **[2]**

Q3) a) For an ideal I in a commutative ring R, define the radical of the ideal I. If I and J are ideals of a commutative ring R with $I \subseteq J$ than prove that $\sqrt{I} \subseteq \sqrt{J}$. **[4]**

b) With usual notations prove that $\sqrt{(9)} = \sqrt{(27)} = \sqrt{(3)}$. **[4]**

c) Show that $\dfrac{Q[x]}{(x+2)}$ is a field. **[2]**

P.T.O.

Q4) a) If $f: R \to S$ is a homomorphism of rings then show that inverse image of a prime ideal is a prime ideal in R if both R and S are commutative. **[4]**

b) If m and n are integers then show that $\frac{z}{(m)}$ is a quotient of $\frac{z}{(mn)}$. Hence find the quotients of $\frac{z}{(6)}$ **[4]**

c) Show that field of fraction of Z[i] is Q[i]. **[2]**

Q5) a) If R is a commutative integral domain with 1 and $p \in R$ is a prime then prove that p is an irreducible in R what can you say about the converse? **[4]**

b) Prove that every Euclidean domain has a unity1. **[4]**

c) Give an example of two elements *a, b* in an Euclidean domain R such that $d(a) = d(b)$ but a and b are not associates, where d in the algorithm map. **[2]**

Q6) a) Prove that every Euclidean domain is a PID (with 1). **[4]**

b) Prove or disprove

The ring $R = \{a + b\sqrt{-5} \mid a, b \in z\}$ is an Euclidean domain. **[3]**

c) If $f: R \to R'$ is a homomorphism of R onto R', which is not an isomorphism and if R is a PID, R' is an Integral domain then show that R' is a field. **[3]**

Q7) a) If R[x] is UFD then prove that R is also UFD. **[4]**

b) With usual notation show that $\frac{Q[x]}{(1+x^2)} \simeq Q[i]$. **[4]**

c) Prove or disprove

The polynomial x^4+1 is irreducible over $\mathbb{R}$. **[2]**

Q8) a) State and prove Schur's lemma for simple modules. **[5]**

b) Define **[5]**

i) Torsion module

ii) Torsion free module

Show that for any module M over a commutative integral domain, the quotient $\frac{M}{M_t}$ is torsion free. (M_t-torsion part of M).

Total No. of Questions : 8]

SEAT No. :

P2078

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[4821]-204

M.A./M.Sc.

MATHEMATICS

MT-604: Linear Algebra

(2013 Pattern) (Semester-II) (Credit System)

Time : 3 Hours] ***[Max. Marks : 50***

Instructions to the candidates:

1) *Answer any five questions.*

2) *Figures to the right indicate full marks.*

3) *Use of non-programmable, scientific calculator is allowed.*

Q1) a) Let V be a vector space over F. If V is generated by a set of m elements, then prove that no linearly independent subset of V has more than m elements. **[5]**

b) Find a basis of the subspace of R^4 generated by the vectors $v_1 = (2, 2, 4, 0)$, $v_2=(4, 2, 0, 4)$, $v_3=(2, 4, 6, 8)$, $v_4=(0, 8, 10, 4)$. **[3]**

c) Let $F^{2\times 2}$ be the vector space of all 2×2 matrices over F and $W_1 = \left\{ \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} \middle| y, z \in F \right\}$ show that W_1 is subspace of $F^{2\times 2}$. **[2]**

Q2) a) If V and U are vector spaces over F and $f: V \to U$ is a linear mapping from V onto U, with kernel K then show that $U \cong V/K$. Further, show that there is a one-to-one correspondence between the set of subspace of V containing K and the set of subspaces of U. **[5]**

b) Let $f: R^2 \to R^2$ be a linear mapping, where $f(a,b) = (2a-b, 4a+5b)$. Find a basis for a range of f and hence determine the rank of f. **[3]**

c) Let $f: R^3 \to R^3$ be defined by $f(x, y, z) = (2x, y-3z, 1)$. Determine whether f is linear transformation. **[2]**

P.T.O.

Q3) a) Let $f: F^{n\times n} \to F^{n\times n}$ be a mapping such that $f(A) = AB$, $A \in F^{n\times n}$ and B is fixed $n \times n$ matrix. **[5]**

i) Prove that f is a linear mapping

ii) Show that ker f = (0) if and only if B is invertible.

b) Let $f: R^3 \to R^3$ be a linear mapping defined by f(a, b, c) = (a, a+b, 0). Find the matrices A and B respectively of the linear mapping f with respect to the standard basis (e_1, e_2, e_3) and the basis (e_1', e_2', e_3') where $e_1' = (1,1,0)$, $e_2' = (0,1,1)$, $e_3' = (1,1,1)$. **[3]**

c) What is the dimension of the vector space V = {P_n- polynomial of degree $\leq$ n, with real coefficients)}. **[2]**

Q4) a) If $\phi \in$ Hom (V, V) and suppose $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are distinct eigenvalues of ϕ and $x_1, \ldots\ldots, x_n$ are eigenvectors associated with $\lambda_1, \ldots\ldots, \lambda_n$ respectively. Then show that the set $\{x_1, x_2, \ldots, x_n\}$ is a linearly independent set. **[5]**

b) The three eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ of a 3×3 matrix A are associated respectively with eigenvalues 1,-1, and 0. Find matrix A. **[3]**

c) Define: **[2]**

i) Orthonormal basis of vector space

ii) Eigen vector

Q5) a) Reduce the following matrix into triangular form $A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -1 & 2 \\ 0 & -3 & 4 \end{bmatrix}$ **[5]**

b) Find the Jordan canonical form of $A = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ **[3]**

c) Determine whether the given set of vectors are orthogonal. $S = \{(1,0,1), (1,0,0), (0,-1,0)\}$ **[2]**

Q6) a) Let V be a vector space of dimension n over F. Then show that there is a 1-1 correspondence between the set of bilinear form on V and the set of n×n matrices over F. **[5]**

b) If B is symmetric bilinear form on a vector space V over a field F and let char (F) $\neq 2$ then prove that there exist an orthogonal basis of V relative to B. **[3]**

c) If the matrix $A=\begin{bmatrix} 4 & 2 & -2 \\ 2 & 1 & 4 \\ -2 & 4 & -2 \end{bmatrix}$ then find quadratic form of the matrix A. **[2]**

Q7) a) Prove that, if T is a self-adjoint operator on a finite-dimensional Euclidean vector space E then there is an orthonormal basis E consisting of eigen vectors of T. **[5]**

b) Let V be the vector space of continuous real valued functions on the interval [0,1]. Define $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$ Show that <,> is a symmetric bilinear form. **[5]**

Q8) a) State and prove Sylvester's theorem. **[5]**

b) If matrix $A=\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 4 \\ -1 & -1 & -2 \end{bmatrix}$ then **[5]**

find a matrix P such that $P^{-1}AP=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Total No. of Questions : 8]

SEAT No. :

P2079

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[4821]-205

M.A. / M.Sc.

MATHEMATICS

MT-605 : Partial Differential Equations

(2013 Pattern) (Semester-II) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Eliminate the parameters a and b from the equation $z = (x^2 + a)(y^2 + b)$. **[3]**

b) Explain the classification of first order partial differential equations as linear, semilinear and quasilinear partial differential equation. **[3]**

c) Find the general solution of $xp + yq = z$. **[4]**

Q2) a) Explain Charpit's method for finding complete integral of a first order PDE $f(x, y, z, p, q) = 0$. **[5]**

b) Explain the method of solving the first order PDE, $g(x, p) = h(y, q)$. **[3]**

c) Write Jacobi's auxiliary equations for $f(x, y, z, u_x, u_y, u_z) = 0$. **[2]**

Q3) a) Find the integral surface of the equation $x^3 p + y(3x^2 + y) q = z(2x^2 + y)$ which passes through the curve $C : x_0 = 1, y_0 = s, z_0 = s(1+s)$. **[5]**

b) Show that $p^2 + q^2 = 1$ and $(p^2 + q^2)x = pz$ are compatible equations. **[3]**

c) Verify that the equation $yzdx + xzdy + xydz = 0$ is integrable. **[2]**

P.T.O.

Q4) a) Derive Analytic expression for the Monge cone at (x_0, y_0, z_0). **[2]**

b) If an element $(x_0, y_0, z_0, p_0, q_0)$ is common to both an integral surface $z = z(x, y)$ and a characteristic strip, then show that the corresponding characteristic curve lies completely on the surface. **[5]**

c) Prove that the solution of Neumann problem is unique up to the addition of a constant. **[3]**

Q5) a) Reduce $\frac{\partial^2 u}{\partial x^2} = (1+y)^2 \frac{\partial^2 u}{\partial y^2}$ to Canonical form. **[5]**

b) Find a complete integral of the following equation by Jacobi's method $(p^2 x + q^2 y) - z = 0$. **[3]**

c) State Harnack's theorem. **[2]**

Q6) a) Using D'Alemberts solution of infinite string find the solution of

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{C^2}\frac{\partial^2 y}{\partial t^2}, 0 < x < \infty, t > 0$$

$$y(x, 0) = u(x),\ y_t(x, 0) = v(x), u \geq 0$$

$$y(0, t) = 0, t \geq 0.$$ **[5]**

b) Prove that the solution of following problem exist then it is unique:

$$u_{tt} - c^2 u_{xx} = f(x, t), 0 < x < l, t > 0$$

$$u(x, 0) = f(x), 0 \leq x \leq l$$

$$u_t(u, 0) = g(x)$$

$$u(0, t) = u(l, t) = 0, t \geq 0.$$ **[5]**

Q7) a) Find the characteristic lines of the hyperbolic equation

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0 .$$ **[5]**

b) Solve the following diffusion equation using Fourier transform technique:

$$u_t = Ku_{xx}, -\infty < x < \infty, t > 0$$

$$u(x, 0) = f(x), -\infty < x < \infty$$ **[5]**

Q8) a) State and prove Kelvin's inversion theorem. **[5]**

b) Prove that the solution for the Dirichlet problem for a circle of radius '*a*' is given by the Poisson integral formula

$$u(\rho, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-\rho^2)\, f(J)\, dJ}{1 - 2\rho\cos(\theta - J) + \rho^2} .$$ **[5]**

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Total No. of Questions : 8]

SEAT No. :

P2080

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[4821]-301

M.A. / M.Sc.

MATHEMATICS

MT-701 : Combinatorics

(2013 Pattern) (Semester-III) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Find a recurrence relation to count the number of n-digit binary sequences with at least one instance of consecutive zeros. **[5]**

b) How many ways are there to pick two different cards from a standard 52-card deck such that

i) The first card is an Ace and the second card is not a Queen?

ii) The first card is a spade and the second card is not a Queen? **[3]**

c) Find a generating function for the number of integer solutions of

$2x+3y+7z=r$ with $x, y, z \geq 0$. **[2]**

Q2) a) How many arrangements of the letters in MISSISSIPPI in which

i) The M is immediately followed by an I?

ii) The M is beside an I (that is, an I is just before or just after the M). **[5]**

b) Solve the recurrence relation $a_n = 3a_{n-1} + n^2 - 3$, with $a_0 = 1$. **[3]**

c) Find two different chessboards (not row or column rearrangements of one another) that have the same rook polynomial. Also, state the corresponding rook polynomial. **[2]**

P.T.O.

Q3) a) How many arrangements of letters in STATISTICS have all of the following properties. **[5]**

i) NO consecutive S's,

ii) Vowels in alphabetical order, and

iii) The 3 T's are consecutive.

b) Solve the recurrence relation.

$a_n = -na_{n-1} + n!$ with $a_0 = 1$. **[3]**

c) How many integer solutions are there to $x_1 + x_2 + x_3 = 0$ with $x_i \geq -5$? **[2]**

Q4) a) How many ways are there to paint the 10 identical rooms in a hotel with five colors if at most three rooms can be painted green, at most three painted blue, at most three red and no constraint on the other two colors, black and white? **[5]**

b) How many 6-letter sequences are there with at least 3 vowels (A, E, I, O, U), if no repetitions are allowed? **[3]**

c) A bookcase has 200 books, 70 in French and 100 about mathematics. How many non-French books not about mathematics are there if there are 30 french mathematics books? **[2]**

Q5) a) How many 10 letter words are there in which each of the letters *e, n, r, s* occur? **[5]**

i) At most once?

ii) At least once?

b) Verify the identity by a committee selection model

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots\cdots\cdots + \binom{n}{n}^2 = \binom{2n}{n}$$ **[3]**

c) Solve the recurrence relation

$a_n = 2a_{\frac{n}{2}} + 2, n \geq 4$ with $a_2 = 1$

(Assume that *n* is a power of 2) **[2]**

Q6) a) Using Generating functions, solve the recurrence relation

$$a_n = a_{n-1} + n(n-1) \text{ with } a_0 = 1.$$ **[5]**

b) If \$500 is invested in a saving account earning 8 percent a year, give a formula for the amount of money in the account after n years. **[3]**

c) Find a generating function for the number of selections of r sticks of chewing gum chosen from eight flavors if each flavor comes in packets of five sticks. **[2]**

Q7) a) Find a ordinary generating function for $a_r = (r-1)^2$. **[5]**

b) How many arrangements are there of TAMELY with either T before A, or A before M, or M before E?

(By 'before', we mean anywhere before, not just immediately before). **[5]**

Q8) a) How many ways are there to send seven different birthday cards, denoted C_1, C_2, C_3, C_4, C_5, C_6, C_7 to seven friends, denoted F_1, F_2, F_3, F_4, F_5, F_6, F_7; if friend F_1 would not like cards C_1 or C_3; if friend F_2 would not like cards C_1 or C_5; if F_4 would not like C_3 or C_6; if F_5 would not like cards C_2 or C_7; if F_7 would not like card C_4 and if friends F_3 and F_6 would like all the cards? **[5]**

b) How many ways are there to arrange the letters in INTELLIGENT with at least two consecutive pairs of identical letters? **[5]**

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Total No. of Questions : 8]

SEAT No. :

P2081

[Total No. of Pages : 3

[4821]-302

M.A. / M.Sc.

MATHEMATICS

MT-702 : Field Theory

(2013 Pattern) (Semester-III) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Let $p(x) \in F[x]$ be an irreducible polynomial of degree 'n' over the field F and $K = F[x]/\langle p(x)\rangle, \theta = x \bmod(p(x))$ in K, then prove that the set $\{1, \theta, \theta^2, \cdots, \theta^{n-1}\}$ form a basis for K as a vector space over F. **[4]**

b) Show that $p(x) = x^3 + 9x + 6 \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$. Find the inverse of $(1 + \theta)$ in $\mathbb{Q}(\theta)$, where θ is a root of $p(x)$. **[4]**

c) Show that the polynomial $f(x) = x^{P^n} - x \in F_P[x]$ is separable. **[2]**

Q2) a) Prove that the extension K/F is finite if and only if K is generated by a finite number of algebraic elements over F. **[4]**

b) If α be an algebraic over F, then prove that there exist a unique monic irreducible polynomial $m_{\alpha, F}(x)$ in F[x] which has α as a root. **[4]**

c) Determine the minimal polynomial over $\mathbb{Q}$ for the element $\sqrt{-1+\sqrt{2}}$. **[2]**

Q3) a) Determine the splitting field and it's degree over Q for the polynomial $x^4 - 2$. **[5]**

P.T.O.

b) Prove that a polynomial $f(x) \in F[x]$ is separable if and only if $f(x)$ and derivative $D_x(f(x))$ are relatively prime. **[3]**

c) State fundamental theorem of Galois theory. **[2]**

Q4) a) Prove that the extension K/F is Galois if and only if K is the splitting field of some separable polynomial over F. **[5]**

b) Show that the composite field of $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt[3]{2})$ is $\mathbb{Q}(\sqrt[6]{2})$, hence determine the degree of composite field $\mathbb{Q}(\sqrt[6]{2})$ over $\mathbb{Q}$. **[3]**

c) Show that $\mathbb{Q}(\sqrt{2}+\sqrt{3})=\mathbb{Q}(\sqrt{2},\sqrt{3})$. **[2]**

Q5) a) Prove that for any field F, if $f(x) \in F[x]$ then there exists an extension K of F, which is a splitting field for $f(x)$. **[5]**

b) Prove or disprove:

"A Galois extension of a Galois extension is Galois". **[3]**

c) If F is any field and $\overline{F}$ is an algebraic closure of F then prove that $F=\overline{F}$ if and only if F is algebraically closed. **[2]**

Q6) a) Show that the polynomial $x^4+1 \in \mathbb{Z}[x]$ is irreducible over $\mathbb{Z}$ and reducible over the field F_p for every prime p. **[5]**

b) Define n^{th} cyclotomic polynomial $\phi_n(x)$. Find cyclotomic polynomial $\phi_n(x)$ for $n = 1, 2, 3, 4, 5$. **[3]**

c) Is the Galois group of the polynomial $x^3 - x + 1 = 0$ solvable? Justify. **[2]**

Q7) a) Let F be a field of characteristic not dividing n which contains the n^{th} roots of unity, then prove that the extension $F\left(\sqrt[n]{a}\right)$ for $a \in F$ is cyclic over F of degree dividing n. **[5]**

b) Show that Galois group of $x^3 - 2 \in \mathbb{Q}[x]$ is the symmetric group on three letters. **[5]**

Q8) a) Suppose L/F is a finite extension and K be any subfield of L containing F, $F \subseteq K \subseteq L$ then prove that [K:F] divides [L:F]. **[5]**

b) Show that doubling the cube and squaring the circle are impossible by straightedge and campass. **[5]**

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Total No. of Questions : 8]

SEAT No. : [ ]

P2082

[Total No. of Pages : 2

[4821]-303
M.A. / M.Sc.
MATHEMATICS
MT-703 : Functional Analysis
(2013 Pattern) (Semester-III) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) State and prove the principle of Uniform Boundedness. **[5]**

b) If $\{x_n\}$ is a sequence in Banach space X such that $\sum_{n=1}^{\infty}\|x_n\| < \infty$, then prove that the series $\sum_{n=1}^{\infty} x_n$ converges in X. **[3]**

c) Give an example of a Banach space which is not Hilbert space. **[2]**

Q2) a) Let H be a Hilbert space and M be a closed subspace of H. Prove that $(\mathrm{M}^{\perp})^{\perp} = \mathrm{M}$. **[4]**

b) Give an example of an isometry on a Hilbert space that is not surjective. Justify. **[4]**

c) State Hahn-Banach theorem. **[2]**

Q3) a) For an operator A on a Hilbert space H, if A = A*, then prove that

$$\|\mathrm{A}\| = \sup\{|\langle \mathrm{A}h, h\rangle| : \|h\| = 1\}.$$ **[5]**

b) Let H be a separable Hilbert space with basis $\{e_n\}$. If A is an operator defined by $\mathrm{A}e_n = \frac{1}{n} e_n$. Then show that A is compact. **[3]**

c) Give an example of a orthonormal basis of $L^2[0, 2\pi]$. **[2]**

P.T.O.

Q4) a) If A is a normal operator and λ, μ are distinct eigen values of A, then prove that $\ker(A-\lambda) \perp \ker(A-\mu)$. **[4]**

b) State open mapping theorem and use it to prove that the inverse of an invertible bounded linear map from a Banach space X to a Banach space Y is bounded. **[4]**

c) Give an example of a (non-identity) unitary operator on an infinite dimensional Hilbert space. **[2]**

Q5) a) Let X be a normed space and f be a linear functional on X. If $\ker f$ is closed, prove that f is continuous. **[3]**

b) Give an example of a convex set in a Hilbert space. **[3]**

c) For any operator A, prove that $(\ker A) = (\mathit{rank}\ A^*)^{\perp}$. **[4]**

Q6) a) State and prove closed graph theorem. **[5]**

b) For $\phi \in L^{\infty}$, let $M_{\phi} : L^2[-\pi, \pi] \to L^2[-\pi, \pi]$ be the operator $M_{\phi}(f) = \phi f$. Find its adjoint $M_{\phi}^{\#}$. Check whether M_{ϕ} is normal. Find the condition on ϕ so that M_{ϕ} is: **[5]**

i) hermitian, ii) unitary.

Show that $M_{\phi}^2 = M_{\phi}$ if and only if ϕ is the characteristic function.

Q7) a) Prove that the operator T is normal if and only if its real and imaginary parts commute. **[4]**

b) If T is a compact self-adjoint operator on a Hilbert space, then prove that either $\pm\|T\|$ is an eigen value of T. **[4]**

c) Define a reflexive space and give an example. **[2]**

Q8) a) State and prove Riesz Representation theorem for a Hilbert space. **[5]**

b) Prove that the identity operator on an infinite dimensional Hilbert space is not compact. **[3]**

c) True or False? Justify. If H is a $\mathbb{R}$-Hilbert space and $\langle Ah, h\rangle = 0$ for all $h \in H$ then $A^* = A$. **[2]**

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Total No. of Questions : 8]

SEAT No. :

P2083

[Total No. of Pages : 2

[4821]-401
M.A. / M.Sc.
MATHEMATICS
MT-801 : Number Theory
(2013 Pattern) (Semester-IV) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Let p be a prime. Then prove that $x^2 \equiv -1 \pmod{p}$ has solutions if and only if $p = 2$ or $p \equiv 1 \pmod{4}$. **[5]**

b) Show that 1763 is composite. **[3]**

c) Prove that $ax + by = a + c$ is solvable if and only if $ax + by = c$ is solvable. **[2]**

Q2) a) Let a, b and $m > 0$ be given integers and $g = (a, m)$. Prove that the congruence $ax \equiv b (\text{mod}\, m)$ has a solution if and only if g/b. **[5]**

b) Find all solutions of the congruence $15x \equiv 25 \pmod{35}$. **[3]**

c) Evaluate $\sum_{j=1}^{\infty} \mu(j!)$. **[2]**

Q3) a) For every positive integer n, prove that $\sum_{d/n} \phi(d) = n$. **[5]**

b) If p and q are distict primes of the form 4K + 3 and $x^2 \equiv p (\text{mod}\, q)$ has no solution. Then prove that $x^2 \equiv q (\text{mod}\, p)$ has two solutions. **[3]**

c) Find all roots of the congruence $x^2 + x + 7 \equiv 0 \pmod{15}$. **[2]**

P.T.O.

Q4) a) Suppose for any odd prime p, let $(a, p) = 1$. Consider the integers $a, 2a, 3a, \cdots \frac{(p-1)}{2}a$ and their least positive residues modulo p. If n denotes the number of these residues that exceed $\frac{p}{2}$, then prove that $\left(\frac{a}{p}\right) = (-1)^n$. **[5]**

b) Find the highest power of 2 dividing 533!. **[3]**

c) Find real numbers x such that **[2]**

i) $[x+3] = 3+x$

ii) $[9x] = 9$.

Q5) a) If P and Q are odd positive integers, and if (P, Q) = 1, then prove that

$$\left(\frac{P}{Q}\right)\left(\frac{Q}{P}\right) = (-1)^{\left\{\frac{P-1}{2} \times \frac{Q-1}{2}\right\}}.$$ **[5]**

b) Exhibit a complete residue system modulo 17 composed entirely of multiples of 3. **[3]**

c) Let P be an odd prime. If there is an integer x such that $P \mid x^2 + 1$, then prove that $P \equiv 1 \pmod 4$. **[2]**

Q6) a) Prove that the product of two primitive polynomials is primitive. **[5]**

b) Prove that $(1 + i)$ is a prime in $Q(i)$. **[3]**

c) Find the smallest integer x such that $d(x) = 6$. **[2]**

Q7) a) Find the least positive integer x such that $x \equiv 5 \pmod 7$, $x \equiv 7 \pmod{11}$ and $x \equiv 3 \pmod{13}$. **[5]**

b) Prove that if ξ is an algebraic number of degree n, then every number in $Q(\xi)$ can be written uniquely in the form $a_0 + a_1\xi + \cdots + a_{n-1}\xi^{n-1}$ where a_i are rational numbers. **[5]**

Q8) a) State and prove Gaussian reciprocity law. **[5]**

b) Find all solutions of $147x + 258y = 369$. **[5]**

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Total No. of Questions : 8]

SEAT No. :

P2084

[Total No. of Pages : 3

[4821]-402

M.A. / M.Sc.

MATHEMATICS

MT-802 : Differential Geometry

(2013 Pattern) (Semester-IV) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Show that graph of a function defined on subset of $\mathbb{R}^n$ is n-surface in $\mathbb{R}^{n+1}$ (function is smooth). **[4]**

b) Show that gradient of smooth function f at $P \in f^{-1}(c)$ is orthogonal to all vectors tangent at f^{-1} (C). **[4]**

c) Sketch the level set and graph of the function $f(x, y) = -x^2 + y^2$. **[2]**

Q2) a) State and prove Lagranges Multiplier theorem. **[5]**

b) Is Mobius band a n-surface? Justify? **[3]**

c) Sketch the vector field on $\mathbb{R}^2$ for $\bar{X}(P) = (P, X(P))$ where X(P) = (0, 1). **[2]**

Q3) a) Find velocity, accelaration and the speed of the parametrized curve $\alpha(t) = (\cos t, \sin t, 2\cos t, 2\sin t)$. **[4]**

b) Show that a parametrized curve α in the unit n-sphere $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$ is geodisic if it is of the form $\alpha(t) = (\cos at)\, e_1 + (\sin at) e_2$ for some orthogonal pairs of unit vectors $\{e_1, e_2\}$ in $\mathbb{R}^{n+1}$ and some $a \in \mathbb{R}$. **[3]**

c) Define Levi-Civita parallelism along with its properties. **[3]**

Q4) a) Show that Gauss map maps a compact connected oriented n-surface in $\mathbb{R}^{n+1}$ onto the unit sphere S^n. **[5]**

b) Show that parallel transport from one tangent space to another along a smooth curve is vector space isomorphism. **[3]**

c) Find the gradient field of the function $f(x, y) = \frac{x^2 + y^2}{4}$. **[2]**

Q5) a) Show that Weingarten map Lp is self adjoint. **[5]**

b) Compute $\nabla_v f$ where $f : \mathbb{R}^{n+1} \to \mathbb{R}, n = 2$ and $v \in \mathbb{R}^{n+1}_{P}$, $p \in \mathbb{R}^{n+1}$ are given by $f(x_1, x_2, x_3) = x_1\, x_2\, x_3^2, \bar{v} = (1, 1, 1, a, b, c)$. **[3]**

c) State inverse function theorem for n-surfaces. **[2]**

Q6) a) Show that there exists a global parametrization for the connected oriented plane curve. **[5]**

b) Show that on each compact n-surface S in $\mathbb{R}^{n+1}$, there exists a point P such that second fundamental form of P is definite (S is oriented). **[5]**

Q7) a) Calculate the Gaussian curvature of the ellipsoid $S : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$ oriented by its outward normal. **[5]**

b) Compute the Weingarten map for the hyperplane

$$a_1\, x_1 + a_2\, x_2 + \cdots + a_{n+1}\, x_{n+1} = b \;\; \left[(a_1, a_2, \cdots, a_{n+1}) \neq (0, 0, \cdots, 0)\right].$$ **[3]**

c) Define Gauss map with atleast one example. **[2]**

Q8) a) Compute the curvature of the circle in $\mathbb{R}^2$ of radius 4 passing through (5, 6) oriented by outward normal $\nabla f / \|\nabla f\|$. **[5]**

b) Show that set of vectors tangent to $f^{-1}(c)$ at p, where p is regular point for the smooth function $f : \mathrm{U} \to \mathbb{R}$ is equal to $[\nabla f(p)]^{\perp}$. **[3]**

c) State second derivative test for local minima / maxima. **[2]**

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Total No. of Questions : 8]

SEAT No. :

P2085

[Total No. of Pages : 4

[4821]-403
M.A. / M.Sc.
MATHEMATICS
MT-803 : Fourier Series and Boundary Value Problems
(2013 Pattern) (Semester-IV) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Suppose that a function $g(u)$ is piecewise continuous on the interval $0 < u < \pi$ and that the right-hand derivative $g_R'(0)$ exists. Then prove that

$$\lim_{N\to\infty}\int_0^{\pi} g(u)\mathrm{D}_{\mathrm{N}}(u)du = \frac{\pi}{2}g(0+),$$

where $\mathrm{D}_{\mathrm{N}}(u)$ is the Dirichlet kernel. **[4]**

b) Find the Fourier cosine series on the interval $0 < x < \pi$ that corresponds to the function $f(x) = \pi - x$. Use this Fourier series to show that

$$\frac{c}{4} - x \sim \frac{2c}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(4n-2)\pi x}{c}, \left(0 < x < \frac{c}{2}\right).$$ **[4]**

c) Find the Fourier sine series on the interval $0 < x < \pi$ that corresponds to the function f defined by

$$f(x) = \begin{cases} 1, \ when\ 0 < x < \dfrac{\pi}{2} \\ 0, \ when\ \dfrac{\pi}{2} < x < \pi \end{cases}.$$ **[2]**

P.T.O.

Q2) a) Let f denote a function that is piecewise continuous on the interval $-\pi < x < \pi$ and periodic, with period 2π, on the entire x-axis. Prove that its Fourier series converges to the mean value

$$\frac{f(x+)+f(x-)}{2}$$

of the one-sided limits of f at each point x $(-\infty < x < \infty)$ where both of the one-sided derivatives $f_R'(x)$ and $f_L'(x)$ exist. **[5]**

b) Find the Fourier series on the interval $-\pi < x < \pi$ that corresponds to the function

$$f(x) = x + \frac{1}{4}x^2 (-\pi < x < \pi).$$ **[3]**

c) Let,

$$f(x) = \frac{e^x - 1}{x}, x \neq 0$$

Find $f(0+)$ and $f_R'(0)$. **[2]**

Q3) a) Let f be a function that is piecewise continuous on the interval $-\pi < x < \pi$.

Then prove that $\int_{-\pi}^{x} f(s)ds = \frac{a_0}{2}(x+\pi) + \sum_{n-1}^{\infty} \frac{1}{n}\left[a_n \sin nx - b_n\left[\cos nx + (-1)^{n+1}\right]\right]$,

$(-\pi \leq x \leq \pi)$. **[5]**

b) Solve the following linear boundary value problem.

$$u_t(x,t) = k\, u_{xx}(x,t) \qquad (0 < x < c, t > 0)$$

$$u_x(0,t) = 0, u_x(c,t) = 0 \qquad (t > 0)$$

$$u(x,0) = f(x) \qquad (0 < x < c)$$ **[5]**

Q4) a) Solve the following boundary value-problem.

$$u_t(x,t) = k\,u_{xx}(x,t) \qquad (0 < x < c, t > 0)$$

$$u(0,t) = 0, u(c,t) = 0, u(x,0) = f(x)$$ **[4]**

b) Let the faces of a plate in the shape of a wedge $0 \le \rho \le a, 0 \le \phi \le \alpha$ be insulated. Find the steady temperatures $u(\rho, \phi)$ in the plate when $u = 0$ on the two rays $\phi = 0, \phi = \alpha\ (0 < \rho < a)$ and $u = f(\phi)$ on the arc $\rho = a(0 < \phi < \alpha)$. Assume that f is piecewise smooth, u is bounded and $\nabla^2 u = 0$. **[4]**

c) If L is the linear operator $L = a^2\dfrac{\partial^2}{\partial x^2} - \dfrac{\partial^2}{\partial t^2}$ and $y_n = \sin\dfrac{n\pi x}{c}\cdot\cos\dfrac{n\pi at}{c}$, then show that $Ly_n = 0 (n = 1, 2, \cdots)$. **[2]**

Q5) a) If $c_n (n = 1, 2, \cdots)$ are the Fourier constants for a function f in $c_p(a,b)$ with respect to an orthonormal set in that space, then prove that $\lim_{n\to\infty} c_n = 0$. **[4]**

b) Derive the eigenvalues and eigenfunctions of the Sturm-Liouville problem $X''(x) + \lambda X(x) = 0, X'(0) = 0, X(c) = 0$. **[4]**

c) Determine the constants A and B such that the function $\varphi_3(x) = 1 + Ax + Bx^2$ is orthogonal to both $\varphi_1(x) = 1$ and $\varphi_2(x) = x$ on the interval $-1 < x < 1$. **[2]**

Q6) a) Let λ be an eigenvalue of the regular Sturm-Liouville problem

$$(rX')' + (q + \lambda p)X = 0\ (a < x < b),$$

$$a_1X(a) + a_2X'(a) = 0, b_1X(b) + b_2X'(b) = 0.$$

If the conditions $q(x) \le 0\ (a \le x \le b)$ and $a_1\ a_2 \le 0, b_1\ b_2 \ge 0$ are satisfied, then prove that $\lambda \ge 0$. **[5]**

b) The boundary value problem $y_{tt}(x,t) = a^2 y_{xx}(x,t) + \mathrm{A}x\sin wt (0 < x < c, t > 0)$, $y(0,t) = y(c,t) = 0,\ y(x,0) = y_t(x,0) = 0$ describes transverse displacement in a vibrating string. Show that resonance occurs when w has one of the values $w_n = \frac{n\pi a}{c}(n = 1, 2, \cdots)$. **[5]**

Q7) a) Solve the following boundary value problem

$$u_t = k\left(u_{\rho\rho} + \frac{1}{\rho}u_\rho\right) \qquad (0 < \rho < c, t > 0)$$

$$u(c,t) = 0 \qquad (t > 0)$$

$$u(\rho,0) = f(\rho) \qquad (0 < \rho < c)$$ **[5]**

b) Verify that the function

$$\mathrm{J}_v(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\,\overline{|}(v+k+1)}\left(\frac{x}{2}\right)^{v+2k} \quad (v > 0)$$

Satisfies Bessel's equation

$$x^2 y''(x) + xy'(x) + (x^2 - v^2)y(x) = 0 \ (v > 0;\ v \neq 1, 2\cdots)$$ **[5]**

Q8) a) Solve the following Legendre's equation $\left[\left(1-x^2\right)y'(x)\right]' + \lambda y(x) = 0$. **[5]**

b) Establish the following properties of Legendre polynomials. **[5]**

i) $\mathrm{P}_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$.

ii) $\mathrm{P}'_{2n}(0) = 0$.

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