Total No. of Questions : 8]

SEAT No. :

P1018

[Total No. of Pages : 2

[5428]-101
M.A./M.Sc. (Semester - I)
MATHEMATICS
MT-501 : Real Analysis
(2013 Pattern)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Define the term support of a function. Also if, $f(x)=x^2-1$ $x\varepsilon$ [0, 2], find supp. $(f(x))$. **[4]**

b) Explain the construction of cantor set.Give any two properties of cantor set. **[4]**

c) Define the terms. **[2]**

i) Measurable function

ii) Simple function

Q2) a) State and prove Monotone Convergence theorem. **[5]**

b) Show that a curve $(x(t), y(t), t\varepsilon[a, b]$ is rectifiable if and only if both $x(t)$ and $y(t)$ are of bounded variation. **[3]**

c) Explain the term "jump discontinuity with an example. **[2]**

Q3) a) State and prove Fatou's Lemma. **[5]**

b) State and prove Rising Sun Lemma. **[5]**

Q4) a) Show that every open set G of $\mathbb{R}$ can be written uniquely as countable union of disjoint open intervals. **[4]**

b) Define the term characteristic function and hence calculate $\int^{L} f(x)dx$ where $f(x) = 2\chi_{[0, 2]}^{(x)} + 5\chi_{[3, 4]}^{(x)}$ **[4]**

c) Define the term "Length of parametrized curve", with an example. **[2]**

P.T.O.

Q5) a) If $E \subset \mathbb{R}^d$ and $E=E_1 \cup E_2$ with $d(E_1, E_2)>0$. Show that $M_*(E) = M_*(E_1) + M_*(E_2)$. Also give an example to show that the result is not true if $E_1 \cap E_2 \neq \phi$ **[5]**

b) If A and B are measurable subsets of $\mathbb{R}^d$ with $A \subseteq E \subseteq B$ and $M(A) = M(B)$. Show that E is measurable and $M(A) = M(E) = M(B)$. **[3]**

c) Define the terms. **[2]**

i) Function of Bounded Variation

ii) Rectifiable Curve

Q6) a) State and prove Bounded Convergence Theorem. **[5]**

b) State and prove Triangle Inequality for $L'(\mathbb{R}^d)$ **[3]**

c) Show that every continuous function is measurable. **[2]**

Q7) a) Suppose $\phi(x)$ is non negative, bounded function in $\mathbb{R}^d$ which is supported on standard unit ball. Define

$K_\delta(x) = \frac{1}{\delta^d}\phi\left(\frac{x}{\delta}\right)$, $\delta > 0$ with $\int_{\mathbb{R}^d} \phi(x)dx = 1$. Then Show that $k_\delta(x)$ is good Kernel. **[5]**

b) State and prove Lebesgue differentiation Theorem. **[5]**

Q8) a) If $F(t)$ is increasing and continuous, then prove that $F'(t)$ exist a.e. on [a,b]. More over, $F'(t)$ is measurable, non negative and $\int_a^b f'(x)dx \leq f(b) - f(a)$. **[5]**

b) Show that complement of a measurable set is again measurable set. **[3]**

c) state Lusin's Theorem. **[2]**

Total No. of Questions : 8]

SEAT No. :

P1019

[Total No. of Pages : 3

[5428]-102
M.A./M.Sc. (Semester - I)
MATHEMATICS
MT-502 : Advanced Calculus
(2013 Pattern) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right, indicate full marks.*

Q1) a) Find values of constant a, b and c such that the directional derivative of $f(x,y,z) = axy^2 + byz + cz^2x^3$ at the point $(1,2,-1)$ has a maximum value 64 in a direction parallel to the z-axis. **[5]**

b) Let $\overline{f} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Prove that $\overline{f}$ is continuous at each point $\overline{a}$ in $\mathbb{R}^n$. **[3]**

c) Let $f(x, y) = \sqrt{|xy|}, (x, y) \in \mathbb{R}^2$. Find $\frac{\partial f}{\partial x}$ at origin. **[2]**

Q2) a) State and prove chain rule for derivatives of scalar fields. **[5]**

b) Evaluate the directional derivative of the function $f(x,y,z) = x^2+2y^2+3z^2$ at (1, 1, 0) in the direction of $\overline{i} - \overline{j} + 2\overline{k}$ **[3]**

c) Prove that if a vector field $\overline{f}$ is differentiable at $\overline{a}$, then $\overline{f}$ is continuous at $\overline{a}$. **[2]**

Q3) a) State and prove first fundamental theorem of calculus for line integrals. **[5]**

b) Define the line integral of a vector field. Evaluate the line integral of $\overline{f}(x, y) = (x^2+y^2)\ \overline{i} + (x^2-y^2)\ \overline{j}$, along the path $y = 1-|1-x|$ from the points (0, 0) to (2, 0). **[5]**

P.T.O.

Q4) a) Let $\overline{f}$ be a vector field continuous on an open connected set S in $\mathbb{R}^n$. If the line integral of $\overline{f}$ is zero around every piece wise smooth closed path in S, then prove that the line integral of $\overline{f}$ is independent of path in S. **[4]**

b) Compute the mass of a spring having the shape of the helix whose vector equation is $\overline{\alpha}(t) = a \cos t\ \overline{i} + a \sin t\ \overline{j} + bt\ \overline{k}$, $0 \le t \le 2\pi$, if the density at (x, y, z) is $x^2 + y^2 + z^2$. **[4]**

c) State Inverse function theorem. **[2]**

Q5) a) Let $\overline{f}(x, y) = P(x, y)\ \overline{i} + Q(x, y)\ \overline{j}$ be a vector field which is continuously differentiable on an open simply connected set S in the plane. Prove that $\overline{f}$ is gradient on S if and only if

$\dfrac{\partial P}{\partial Y} = \dfrac{\partial Q}{\partial X}$ everywhere on S. **[5]**

b) Use Green's theorem to evaluate the line integral $\oint_C y^2 dx + x dy$, where C is the square with vertices (0, 0), (2, 0), (2, 2) and (0, 2) oriented counter clockwise. **[3]**

c) Let $x = \rho \cos\theta \sin\phi$, $y = \rho \sin\theta \sin\phi$, $z = \rho \cos\phi$. Find $J(\rho, \theta, \phi)$. **[2]**

Q6) a) Show that $\iint_S f(xy)\,dx\,dy = \log 2 \int_1^2 f(u)\,du$, Where S is the region in the first quadrant bounded by the curves $xy=1$, $xy=2$, $y = x$, $y = 4x$. **[5]**

b) Evaluate $\iiint_S \sqrt{x^2 + y^2}$ dxdydz, where S is the solid formed by the upper nappe of the cone $z^2 = x^2+y^2$ and the plane $z=1$. **[5]**

Q7) a) *A* parametric surface *S* is described by the vector equation $\overline{r}(u,v) = u\cos v\ \overline{i} + u\sin v\ \overline{j} + u^2\ \overline{k}$, where $0 \le u \le 4$ and $0 \le v \le 2\pi$. find area of surface *S*. **[5]**

b) Define fundamental vector product of a parametric surface. Find the fundamental vector product of the surface $\overline{r}(x,y) = x\,\overline{i} + y\,\overline{j} + f(x,y)\,\overline{k}$. **[3]**

c) State stokes theorem. **[2]**

Q8) a) State and prove Gauss divergence theorem. **[5]**

b) Let $\overline{F}(x,y,z) = y^2\overline{i} + xy\overline{j} + xz\overline{k}$, let S be the hemisphere $x^2+y^2+z^2 = 1$, $z \geq 0$ and $\overline{n}$ be the unit normal with a non-negative z-component. Use stokes theorem to transform the surface integral $\iint_s (curl\overline{F}).\overline{n}\,ds$ to a line integral and then evaluate the line integral. **[5]**

Total No. of Questions : 8]

SEAT No. :

P1020

[Total No. of Pages : 2

[5428]-103
M.A./M.Sc. (Semester - I)
MATHEMATICS
MT - 503 : Group Theory
(2013 Pattern) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) What are the symmetries of a square? Hence find dihedral group of order 8 with the Cayley table. **[5]**

b) Prove that in a group G, there is only one identity element. **[3]**

c) Give two reasons, why the set of odd integers under addition is not a group. **[2]**

Q2) a) State and prove one step subgroup test. **[5]**

b) List all subgroups of z_{30}. **[3]**

c) Show that $U(14) = < 3 > = < 5 >$ **[2]**

Q3) a) Let $G = < a >$ be a cyclic group of order n. Then prove that $G = < a^k >$ if and only if g.c.d.$(k, n) = 1$. **[4]**

b) Show that every permutation can be written as a product of disjoint cycles. **[4]**

c) What is the order of the following permutations? **[2]**

i) $\begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 2\,1\,5\,4\,6\,3 \end{pmatrix}$

ii) $\begin{pmatrix} 1\,2\,3\,4\,5\,6\;7 \\ 7\,6\,1\;2\,3\,4\,5 \end{pmatrix}$

P.T.O.

Q4) a) State and prove fundamental theorem of cyclic groups. **[5]**

b) State and prove orbit-stabilizer theorem. **[5]**

Q5) a) Define Inner automorphism induced by an element of group G and show that it forms a group under the operation, composition of functions. **[4]**

b) Suppose K is a proper subgroup of H and H is a proper subgroup of G. If $|K| = 42$ and $|G| = 420$, what are the possible orders of H? **[4]**

c) Compute 5^{15} modulo 7. **[2]**

Q6) a) Suppose s and t are relatively prime. Then prove that $U(st)$ is isomorphic to the external direct product of $U(s)$ and $U(t)$. **[5]**

b) Define factor group and list the elements of $z/<20>$ and $<4>/<20>$. **[3]**

c) Verify Cayley's theorem for $U(12)$. **[2]**

Q7) a) Let G be a group of order pq, where p, q are primes, $p<q$ and p does not divide $q-1$. Then prove that G is cyclic. **[5]**

b) Let ϕ be a group homomorphism from $G \to \overline{G}$. Then prove that ker. ϕ is a normal subgroup of G. **[3]**

c) State Greedy algorithm for an Abelian group of order p^n. **[2]**

Q8) a) Determine all the groups of order 99. **[5]**

b) Show that a group of rotations of a cube is isomorphic to D_4. **[5]**

Total No. of Questions : 8]

SEAT No. :

P1021

[Total No. of Pages : 3

[5428]-104
M.A./M.Sc. (Semester - I)
MATHEMATICS
MT-504 : NumericaL Analysis
(2013 Pattern) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) ***Solve any five questions out of eight questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) Let g be continuous on the closed interval [a, b] with $g : [a, b] \to [a, b]$. Then g has fixed point $p \in [a, b]$. Furthermore, if g is differentiable on the open interval (a, b) and there exists a positive integer $k < 1$ such that $| g'(x) | \leq k < 1$ for all x in (a, b), then show that fixed point in $[a, b]$ is unique. **[5]**

b) Verify that $x = \sqrt{a}$ is a fixed point of the function $g(x) = \frac{x^3 + 3xa}{3x^2 + a}$.

Also determine the order of convergence. **[3]**

c) Compute the following limit and determine the corresponding rate of convergence $\lim_{n \to 0} \frac{\sin n}{n}$. **[2]**

Q2) a) Show that the function $g(x) = e^{-x^2}$ has unique fixed point on [0, 1] by using fixed point iteration method and starting Value $p_0 = 0$. **[5]**

b) The function $f(x) = x^3 + 2x^2 - 3x - 1$ has a root on the interval (1, 2). Approximate this zero within an absolute tolerance of 5×10^{-5} using Newton's method starting with $p_0 = 1$. **[3]**

c) Define the terms :

i) Order of Convergence

ii) The degree of precision. **[2]**

P.T.O.

Q3) a) Solve the following system of equation using Gaussian elimination with scaled partial pivoting. **[5]**

$$2x - y + z = 2$$
$$4x + 2y + z = 7$$
$$6x - 4y + 2z = 4$$

b) Determine the Doolittle decomposition of the given matrix and then solve the system $Ax = b$ for the right hand side vector. **[3]**

$$\text{where } A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \\ 3 & 2 & -1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

c) Compute the condition number K_∞ for the matrix $A = \begin{bmatrix} 1/2 & 1/3 \\ 1/3 & 1/4 \end{bmatrix}$. **[2]**

Q4) a) Show that the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$ has no LU decomposition.

Rearrange the rows of A so that the resulting matrix does have an LU decomposition. **[5]**

b) Solve the following system of linear equations by Jacobi method, start with $x^{(0)} = [0\ 0\ 0]^T$, (Perform 2 iterations). **[3]**

$$5x_1 + x_2 + 2x_3 = 10$$
$$-3x_1 + 9x_2 + 4x_3 = -14$$
$$x_1 + 2x_2 - 7x_3 = -33$$

c) Show that the Matrix $\begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$ has no LU decomposition. **[2]**

Q5) a) For the matrix $A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}$ with initial vector $x^{(0)} = [1\ 0\ 1]^T$ Perform three iteration of power method. **[5]**

b) Define Householder matrix and show that it is symmetric and orthogonal. **[3]**

c) Find the vector valued function F associated with the following system and compute the Jacobain of F. **[2]**

$$x^3 - 20x - y + 15 = 0$$
$$x + xy^2 - 30y + 1 = 0$$

Q6) a) Derive the Open Newton-Cotes formula with n=3;

$$\int_a^b f(x) = \tfrac{b-a}{24}[11f(a+\triangle x)+f(a+2\Delta x)+f(a+3\Delta x)+11f(a+4\Delta x)].$$ **[5]**

b) Derive the following backward difference approximation for the second derivative. **[3]**

$$f''(x_0) \approx \tfrac{f(x_0-2h)-2f(x_0-h)+(x_0)}{h^2}$$

c) If $f(x) = 1 + x + x^3$ find $f'(2)$ for $h = 0.01, 0.001$. **[2]**

Q7) a) Verify that the composite midpoint rule has rate of convergence $O(h^2)$ by approximating the value of $\int_0^1 \sqrt{1+x^3}\,dx$. **[5]**

b) Use Householder's method to reduce the following symmetric matrix to tridaigonal form. **[5]**

$$A=\begin{bmatrix} 4 & 1 & -2 & 1 \\ 1 & 3 & 1 & -1 \\ -2 & 1 & 2 & 0 \\ 1 & -1 & 0 & 5 \end{bmatrix}$$

Q8) a) Apply Euler's method to approximate solution of initial value problem, $\tfrac{dx}{dt} = \tfrac{e^t}{x}, 0 \le t \le 1, x(0) = 1,$ Using 4 steps. Find the corresponding error in each step. **[5]**

b) Find solution of the initial value problem, **[5]**

$\tfrac{dx}{dt} = tx^2 - x, 1 \le t \le 5, x(1) = \tfrac{-1}{ln(2)}$ using fourth order Runge Kutta method with a step size $h = 1$.

Total No. of Questions : 8]

SEAT No. :

P1022

[Total No. of Pages : 3

[5428]-105
M.A./M.Sc. (Semester - I)
MATHEMATICS
MT-505 : Ordinary Differential Equations
(2013 Pattern) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) If $y_1(x)$ and $y_2(x)$ are two solutions of differential equation $y'' + P(x)y' + Q(x)y = 0$ on closed interval [a, b], then prove that their wronskian $W = W(y_1, y_2)$ is either identically zero or never zero on [a, b]. **[5]**

b) Find the general solution of $y'' + 4y = 3$ sinx by using method of undetermined coefficients. **[3]**

c) Show that $y = c_1 \sin x + c_2 \cos x$ is the general solution of $y'' + y = 0$ on any interval. **[2]**

Q2) a) If $y_1(x)$ is one solution of the differential equation $y'' + P(x)\, y' + Q(x)y = 0$, then find the other solution. **[5]**

b) Find the particular solution of $y'' + y = \sec x$ by method of variation of parameters. **[3]**

c) Find the solution of the following initial value problem $y'' - 6y' + 5y = 0$, $y(0) = 3$ and $y'(0) = 11$. **[2]**

Q3) a) State and prove sturm seperation theorem. **[5]**

b) Let u(x) be any nontrivial solution of $u'' + q(x)\, u = 0$ where $q(x) > 0$ for all $x > 0$. If $\int_1^{\infty} q(x)\,dx = \infty$, then prove that $u(x)$ has infinitely many zeros on the positive X-axis. **[3]**

c) Find the normal form of Bessel's equation $x^2 y'' + x y' + (x^2 - p^2)y = 0$. **[2]**

P.T.O.

Q4) a) Find the general solution of $(1+x^2)\,y''+x\,y'+y = 0$ interms of power series of x. **[5]**

b) Find the indicial equation and its roots of the differential equation

$$4x^2\,y'' + (2x^4-5x)\,y' + (3x^2+2)y = 0.$$ **[3]**

c) Locate and classify the singular points on the x-axis

$$x^2\,(x^2-1)\,y''-x\,(1-x)\;y' + 2y = 0.$$ **[2]**

Q5) a) Use the method of Frobenius series to solve the differential equation $2x^2\,y'' + x\,(2x+1)\,y'-y = 0$ about the regular singular point 0. **[5]**

b) Prove that $\sin x = x\left[\lim_{a\to\infty} F\left(a,a,\frac{3}{2},\frac{-x^2}{4a^2}\right)\right]$. Where F is hypergeometric function. **[3]**

c) Find the critical points of

$$\frac{dx}{dt} = y^2 - 5x + 6$$

$$\frac{dy}{dt} = x - y.$$ **[2]**

Q6) a) Find the general solution of the system

$$\frac{dx}{dt} = 7x + 6y$$

$$\frac{dy}{dt} = 2x + 6y.$$ **[5]**

b) determine the nature of the point at $x = \infty$ for the differential equation $x^2\,y''+4x\,y'+2y = 0$. **[3]**

c) Prove that the function $E(x, y) = ax^2 + bxy + cy^2$ is positive definite if and only if $a > 0$ and $b^2-4ac < 0$. **[2]**

Q7) a) If M_1 and M_2 are roots of auxiliary equation of the system **[5]**

$$\frac{dx}{dt} = a_1 x + b_1 y$$

$$\frac{dy}{dt} = a_2 x + b_2 y$$

which are complex conjugate but not pure imaginary, then prove that critical point (0, 0) is a spiral.

b) Show that the function $f(x, y) = y^{1/2}$ satisfies Lipschitz condition on rectangle $|x| < 1$ and $c \le y \le d$ where $0 < c < d$; but it does not satisfy Lipschitz condition on rectangle $|x| \le 1$ and $0 \le y \le 1$. **[5]**

Q8) a) Find the general solution near $x = 0$ of the hypergeometric equation $x(1-x)\, y'' + [c-(a+b+1)]\, y' - aby = 0$ where a, b, c are constants. **[5]**

b) Find the exact solution of the initial value problem $y' = 2x(1+y), y(0) = 0$. starting with $y_0(x) = 0$, calculate $y_1(x)$, $y_2(x)$, $y_3(x)$, $y_4(x)$ by Picard's method. **[5]**

Total No. of Questions : 8]

SEAT No. :

P1023

[Total No. of Pages : 2

[5428]-201
M.A./M.Sc. (Semester - II)
MATHEMATICS
MT - 601 : Complex Analysis
(2013 Pattern) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) ***Attempt any five questions.***
2) ***Figures to the right indicate full marks.***

Q1) a) If a function f is holomorphic at z_0, then prove that

$$\frac{\partial f(z_0)}{\partial \overline{z}} = 0, \text{ and } f'(z_0) = \frac{\partial f(z_0)}{\partial z} = 2\frac{\partial u(z_0)}{\partial z}.$$ **[4]**

b) Show that, in polar coordinates, the Cauchy-Riemann equations take the form $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$. **[3]**

c) Let γ be a smooth curve in $\mathbb{C}$ parametrized by $z : [a, b] \to \mathbb{C}$ and γ^- be γ with the reverse orientation. For any continuous function f prove that

$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz.$$ **[3]**

Q2) a) Consider the convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Prove that the derivative of f is also a power series obtained by differentiating term by term the series for f and f' has the same radius of convergence as f. **[5]**

b) Suppose that U and V are open sets in the complex plane. Prove that if $F : U \to V$ and $g : V \to \mathbb{C}$ are two functions that are differentiable (in the real sense, that is, as functions of the two real variables x and y), and $h = g \circ f$ then **[5]**

$$\frac{\partial h}{\partial \overline{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial \overline{z}}.$$

P.T.O.

Q3) a) Show that it is impossible to define a total ordering on $\mathbb{C}$. **[5]**

b) Let $|z|=1$. For a fixed ω in the unit disc D, prove that the mapping

$$F : z \to \frac{\omega - z}{1 - \bar{\omega} z}$$

satisfies $|F(z)| = 1$. **[3]**

c) Prove that the power series

$$\sum_{n=0}^{\infty} \frac{nz^n}{1+n^3}$$

converges at every point of the unit circle. **[2]**

Q4) a) State and prove Goursat's Theorem. **[5]**

b) Prove that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$. **[3]**

c) Prove that $\sin z$, $z \in \mathbb{C}$ is not bounded. **[2]**

Q5) a) State and prove Schwarz reflection principle. **[5]**

b) Prove that for all $\xi \in \mathbb{C}$, we have $e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x \xi} dx.$ **[3]**

c) State Cauchy inequality. **[2]**

Q6) a) State and prove open mapping theorem. **[5]**

b) Suppose that a function f has a pole at a point $z_0 \in \Omega$, then prove that in a neighborhood of z_0 there exist a non-vanishing holomorphic function h and a unique positive integer n such that $f(z) = (z-z_0)^{-n} h(z)$. **[3]**

c) Give values of $z_1, z_2 \in \mathbb{C}$ such that $\log (z_1 z_2) \neq \log z_1 + \log z_2$. **[2]**

Q7) a) State and prove argument principle. **[5]**

b) Prove that the set $\Omega = \mathbb{C} - \{0\}$ is not simply connected. **[5]**

Q8) a) State and prove maximum modulus principle. **[5]**

b) i) Prove that any holomorphic function in a simply connected domain has a primitive. **[3]**

ii) Find the radius of convergence of the series $\sum_{n=1}^{\infty} (\log n)^2 z^n$. **[2]**

Total No. of Questions : 8]

SEAT No. :

P1024

[Total No. of Pages : 3

[5428]-202
M.A./M.Sc. (Semester - II)
MATHEMATICS
MT-602 : Topology
(2013 Pattern) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right, indicate full marks.*

Q1) a) Let A be a set. Prove that there is no injective map $f : \mathbb{P}(A) \to A$ and there is no surjective map $g : A \to \mathbb{P}(A)$. **[5]**

b) Consider the following topologies on $\mathbb{R}$:

τ_1 : The standard topology;

τ_2 : The topology of $\mathbb{R}_k$;

τ_3 : The finite complement topology;

τ_4 : The upper limit topology, having all sets [a, b] as basis ;

τ_5 : The topology having all sets $(-\infty, a)=\{x|x<a\}$ as basis.

Determine, for each of these topologies, which of the others, it contains. **[3]**

c) Show that the order topology on the set $X = \{1, 2\}\times\mathbb{Z}_+$ in the dictionary order is not the discrete topology. **[2]**

Q2) a) Define the projection maps π_1 and π_2 and prove that the collection $S=\{\pi_1^{-1}(U)\,|\,U \text{ is open in } X\}\cup\{\pi_2^{-1}(V)\,|\,V \text{ is open in } Y\}$ is a sub basis for the product topology on X×Y. **[4]**

b) Let Y be a subspace of X ; let A be a subset of Y ; let $\overline{A}$ denote the closure of A in X. Then prove that the closure of A in Y equals $\overline{A}\cap Y$. **[3]**

c) If X is a hausdorff space, then prove that a sequence of points of X converges to at most one point of X. **[3]**

P.T.O.

Q3) a) Let X and Y be topological spaces; let $f : X \to Y$. Prove that the following statements are equivalent :

i) f is continuous.

ii) for every subset A of X, $f(\bar{A}) \subset \overline{f(A)}$.

iii) for every closed subset B of Y, the set $f^{-1}(B)$ is closed in X. **[5]**

b) Give an example of bijective continuous map that is not a homeomorphism. Justify. **[3]**

c) If U is open in X and A is closed in X, then show that U\A is open in X and A\U is closed in X. **[2]**

Q4) a) Let $f : A \to X \times Y$ be given by the equation $f(a)=(f_1(a), f_2(a))$. then prove that f is continuous if and only the functions $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous. **[5]**

b) State and prove the sequence lemma. **[3]**

c) Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that f(x) is a limit point of f(A)? justify your answer. **[2]**

Q5) a) Define box and product topology. State and prove the comparison of the box and product topologies. **[5]**

b) Let $p : X \to Y$ be a quotient map. Let Z be a space and $g : X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$, for $y \in Y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$. Prove that the induced map f is continuous if and only if g is continuous. Also prove that f is quotient map if and only if g is quotient map. **[5]**

Q6) a) Prove that compactness implies limit point compactness but not conversely. **[4]**

b) Prove that the union of a collection of connected subspaces of X that have a point in common is connected. **[3]**

c) Prove that a space X is locally connected if and only if for every open set U of X, each component of U is open in X. **[3]**

Q7) a) State and prove the tube lemma. **[4]**

b) Prove that the image of a compact space under a continuous map is compact. **[3]**

c) State :

i) The Urysohn lemma.

ii) The Tietze extension theorem.

iii) The Tychonoff theorem. **[3]**

Q8) a) Let X be a topological space. Let one-point sets in X be closed. Prove that X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\bar{V} \subset U$. **[5]**

b) Prove that every metrizable space is normal. **[5]**

Total No. of Questions : 8]

SEAT No. :

P1025

[Total No. of Pages : 2

[5428]-203
M.A./M.Sc. (Semester - II)
MATHEMATICS
MT-603 : Ring Theory
(2013 Pattern) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Suppose R is a ring with 1 such that non-units in R form a subgroup of $(R, +)$, then prove that char (R) is either 0 or else a power of prime number. **[4]**

b) If R is a ring with 1 and I is an ideal in R, then show that there is a maximal ideal M of the same kind I such that $I \subseteq M$. **[4]**

c) Prove or disprove : **[2]**

The ring $\mathbb{Z}_6[x]$ is an integral domain.

Q2) a) Let R be a ring with 1. Prove that R is a division ring if and only if (O) and R are the only left ideals in R. **[5]**

b) In any ring R, show that ab is nilpotent if and only if ba is nilpotent, where $a, b \in R$. **[3]**

c) Give an example of subring S of a ring R in which an element a $\in S$ may be zero divisor in R, but not in S. Justify. **[2]**

Q3) a) For $n \geq 2$, prove that $\mathbb{Z}/n\mathbb{Z}$ has no nontrivial nilpotent elements if and only if n is square free. **[5]**

b) Let P be a prime in $\mathbb{Z}$. Show that $M = \{(x, py) \mid x, y \in \mathbb{Z}\}$ is a maximal ideal of $\mathbb{Z} \times \mathbb{Z}$. **[3]**

c) Prove that in a commutative ring, every maximal ideal is a prime ideal. **[2]**

Q4) a) Let f : R $\rightarrow$ S be a homomorphism of rings. If R and S are commutative rings, then prove that inverse image of a prime ideal is a prime ideal in R. **[5]**

b) Prove or disprove : **[3]**

An element $\overline{5}+\overline{4}x+\overline{6}x^2$ is a unit in $\mathbb{Z}_8$[x].

c) Is $\dfrac{Q[x]}{<x+2>}$ a field ? Justify. **[2]**

Q5) a) Prove that every Euclidean domain is principal ideal domain. **[5]**

b) Prove or disprove :

The ring $R=\left\{a+b\sqrt{-5} \mid a,b\in\mathbb{Z}\right\}$ is an Euclidean domain. **[3]**

c) Show that field of fraction of $\mathbb{Z}$[i] is $\mathbb{Q}$[i]. **[2]**

Q6) a) Let P be an ideal in a ring R. Prove that I is 2-sided ideal in R if and only if I is the kernel of some homomorphism f : R $\rightarrow$ S, for a suitable ring S. **[5]**

b) Let R be an integral domain. Then prove that R[x] is unique factorization domain if and only if R is a unique factorization domain. **[5]**

Q7) a) If the ring End_k (v) is a simple ring, then prove that V is finite dimensional vector space over the field K. **[4]**

b) Let R = C ([0, 1], $\mathbb{R}$] be the ring of all continuous functions defined on [0, 1]. Show that R is an integral domain. **[4]**

c) Give an example of nontrivial commutative ring in which square of every element is zero. **[2]**

Q8) a) Define : **[4]**

i) Free module

ii) Torsion free module.

Further, give an example of torsion free module which is not a free module.

b) Prove that a vector space is a free module. **[4]**

c) Prove or disprove : **[2]**

Any minimal submodule is a simple module.

Total No. of Questions : 8]

SEAT No. :

P1026

[Total No. of Pages : 3

[5428]-204
M.A./M.Sc. (Semester - II)
MATHEMATICS
MT-604 : Linear Algebra
(2013 Pattern) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*
3) *Use of non-programable calculator is allowed.*

Q1) a) Let V and U be vector spaces over F. Let $f : V \to U$ be a linear mapping from V onto U, with kernel K then show that $U \cong V/K$. Further show that there is one-to-one correspondence between the set of subspaces of V containing K and the set of subspaces of U. **[5]**

b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be linear mapping such that $f(1, 2) = (1, -1)$ and $f(2, -3) = (4,1)$ then find $f(a, b)$. **[3]**

c) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_3 - x_1)$ then show that f is a linear mapping. **[2]**

Q2) a) Let U, V be vector spaces over F and Let $f : U \to V$ be linear mapping then show that **[5]**

i) Ker f is a subspace of U.

ii) The range of f is subspace of V.

b) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined by $f(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_1 + x_2 + x_3)$ then find ker f. **[3]**

c) Complete the set $\{ (1, 1, 0, 0), (0, 0, 1, 1), (0, 1, 1, 0)\}$ to form a basis of $\mathbb{R}^4$. **[2]**

P.T.O.

Q3) a) Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. Suppose that V is finite-dimensional then show that rank (T) + nullity (T) = dim V. **[5]**

b) Let $V = F[x]$ be the vector space of polynomials over F in a variable x, and let W be the subspace of polynomials of degree ≤ 4. Find basis of V/W. **[3]**

c) Let Q be linear mapping on R^2 defined by $Q(x_1, x_2) = (-x_2, x_1)$. Write the matrix of Q with respect to the basis (e_1, e_2) where $e_1 = (1, 2)$ and $e_2 = (-1, 1)$. **[2]**

Q4) a) Let A be an n × n matrix, then show that there exists an invertible matrix P such that $P^{-1} A P$ is in Jordan canonical form. **[5]**

b) Find the triangular matrix form of the following matrix : **[3]**

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -1 & 2 \\ 0 & -3 & -4 \end{bmatrix}$$

c) Find eigen vectors of matrix A corresponding to the eigen value $\lambda = 1$;

$$\text{where } A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$ **[2]**

Q5) a) If $A \in \mathbb{F}^{n\times n}$ matrix has n distirct eigen values $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots \lambda_n$ then show that there exists an invertible matrix P such that $P^{-1} AP = \text{diag}(\lambda_1, \lambda_2, ----\lambda_n)$ **[5]**

b) Prove that similar matrices have the same eigen values. **[3]**

c) If A is nonzero idempotent matrix then show that 0, 1 are the only eigen values of A . **[2]**

Q6) a) Reduce the following matrix to its Jordan canonical form :

$$A = \begin{bmatrix} 5 & 1 & -2 & 4 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$ **[5]**

b) If λ, μ are distinct eigen-values of A then show that $\ker (A - \lambda I)^p \subset \ker (A-\mu I)^q$ for all positive integers p and q. **[5]**

Q7) a) Let B be a bilinear form on the vector space $\mathbb{R}^3$ whose matrix with respect to the Standard basis of $\mathbb{R}^3$ is

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

find the matrix B relative to the basis $\{(1, 1, 0), (0, 1, 0), (1, 1, 1)\}$ of $\mathbb{R}^3$ **[5]**

b) Let H be a hermitial form on V. Then show that a linear mapping T of V into itself is H-unitary if and only if $H(T_x, T_x) = H(x, x)\ \forall x \in V$. **[5]**

Q8) a) Reduce the quadratic form $4x^2+y^2-8z^2+4xy-4xz+8yz$ to its diagonal form by an orthogonal transformation of coordinates. **[5]**

b) Let E be a euclidean vector space. Let $x, y \in E$

i) If $x \perp y$ then show that $||x + y||^2 = ||x||^2 + ||y||^2$

ii) Show that $| < x, y > | \leq || x ||\cdot || \, || y ||$. **[5]**

Total No. of Questions : 8]

SEAT No. :

P1027

[Total No. of Pages : 2

[5428] - 205
M.A./M.Sc.
MATHEMATICS
MT - 605 : Partial Differential Equations
(2013 Pattern) (Semester - II) (Credit System)

Time :3 hours] *[Max. Marks :50*

Instructions to the candidates:

1) *Answer any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Find the general integral of $yzp + xzq = xy$. **[4]**

b) Show that $x = a\sin u\cos v, y = a\sin u\sin v,$

$Z = a\cos u$ and $x = a\left(\frac{1-v^2}{1+v^2}\right)\cos u,$

$Y = a\left(\frac{1-v^2}{1+v^2}\right)\sin u,\quad Z = \frac{2av}{1+v^2}$ both gives the spherical surface $x^2 + y^2 + z^2 = a^2$ **[4]**

c) Define the term Quasi - linear equation with an example. **[2]**

Q2) a) Find the general solution of : **[5]**

$(1 + yz)\, dx + [x\, (z - x)]\, dy - (1 + xy)\, dz = 0$

b) Find the complete integral of $Z = p^2 + q^2$ by charpit's method. **[3]**

c) Define curve in space with an example. **[2]**

Q3) a) Find the orthogonal trajectories on the cone $x^2 + y^2 = z^2 \tan^2\alpha$ of its intersection with family of planes parallel to $z = 0$. **[5]**

b) State and prove necessary and sufficient condition for the pfaffian differential equation in 3- dimensions. **[5]**

P.T.O.

Q4) a) Explain the charpit's method of solving non-linear equation $f(x,y,z,p,q) = 0$. **[5]**

b) If $u(x,y,z)$ is a function satisfying the partial differential equation $(y-z)\frac{\partial u}{\partial x}+(z-x)\frac{\partial u}{\partial y}+(x-y)\frac{\partial u}{\partial z}=0$ Show that $u\ (x,y,z)$ is either $x + y + z$ or $x^2 + y^2 + z^2$. **[3]**

c) Define surface in space with example. **[2]**

Q5) a) State and prove kelvin's inversion theorem. **[5]**

b) Show that the differential $2xz + q^2 = x\ (px + qy)$ has complete integral $z + a^2 x = axy + bx^2$ **[3]**

c) Solve Clairaut's equation by using Charpit's method. **[2]**

Q6) a) Explain Jacobi's method of solving partial differential equation $f(x_1, x_2, x_3, p_1, p_2, p_3) = 0$, where $P_i = \frac{\partial z}{\partial x_i}, i = 1, 2, 3$ **[5]**

b) State and prove Harnack's theorem. **[5]**

Q7) a) State Neumann's problem for the upper Half plane and find its solution. **[5]**

b) State and prove Green's theorem. **[3]**

c) Reduce the equation $u_{xx} - x^2 u_{yy} = 0$ to canonical form. **[2]**

Q8) a) If u(x,y) is harmonic in a bounded domain D and continuous in $\bar{D} = D \cup B$, then $u(x,y)$ attains its maximum on the boundary B of D. **[4]**

b) State Dirichlet problem for rectangle. **[2]**

c) Find the condition for the two equations f $(x, y, z, p, q) = 0$ and $g\ (x, y, z, p, q) = 0$ to be compatible. **[4]**

Total No. of Questions : 8]

SEAT No. :

P1028

[Total No. of Pages : 3

[5428]-301
M.A./M.Sc. (Semester - III)
MATHEMATICS
MT-701 : Combinatorics
(2013 Pattern) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) How many ways are there to distribute three different teddy bears and nine identical lollipops to four children : **[5]**

i) Without restriction ?

ii) With no child getting two or more teddy bears?

iii) With each child getting three "goodies"?

b) What is the probability that the sum of two randomly chosen integers between 20 and 40 inclusive is even (the possibility of the two integers being equal is allowed)? **[3]**

c) Find the number of derangements of 1, 2, 3, 4, 5 using the associated chessboard of darkened squares. **[2]**

Q2) a) How many 8-letter arrangements can be formed from the 26 letters of the alphabet (without repetition) that include at most 3 of the 5 vowels and in which the vowels appear in alphabetical order. **[5]**

b) Solve the following recurrence relation assuming that n is a power of 2. (leaving a constant A to be determined.) **[3]**

$$a_n = 4\, a_{n/2} + 3n$$

c) Find the coefficient of x^{25} in $(1 + x^3 + x^8)^{10}$. **[2]**

P.T.O.

Q3) a) How many ways are there to invite one of three different friends over for dinner on six successive nights such that no friend is invited more than three times. **[5]**

b) Solve the recurrence relation. **[3]**

$a_n = 3\, a_{n-1} - 4n + 3(2^n)$, with $a_1 = 8$.

c) Find a generating function for a_r, the number of ways n distinct dice can show a sum of r. **[2]**

Q4) a) Find ordinary generating function whose coefficient a_r equals r(r–1) (r–2) (r–3). **[5]**

Hence, evaluate the sum

$4 \times 3 \times 2 \times 1 + 5 \times 4 \times 3 \times 2 + \ldots\ldots\ldots\ldots.. + n(n-1)\,(n-2)\,(n-3)$

b) How many 10 letter words are there in which each of the letters e, n, r, s occur at most once? **[3]**

c) Show by a combinatorial argument that **[2]**

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

Q5) a) How many ways are there to roll 10 distinct dice so that all 6 faces appear? **[5]**

b) Solve the recurrence relation **[3]**

$a_n = -na_{n-1} + n!$ where $a_0 = 1$

c) Find a generating function for the number of selections of r sticks of chewing gum chosen from eight flavors if each flavor comes in packets of five sticks. **[2]**

Q6) a) Using generating functions, solve the recurrence relation **[5]**

$a_n = 2a_{n-1} + 2^n$ with $a_0 = 1$.

b) How many ways are there to split 6 copies of one book, 7 copies of a second book and 11 copies of a third book between two teachers if each teacher gets 12 books and each teacher gets at least 2 copies of each book. **[3]**

c) Find a generating function for the number of ways to write the integer r as a sum of positive integers in which no integer appears more than three times. **[2]**

Q7) a) Find a recurrence relation for the number of n-digit quaternary (0, 1, 2, 3) sequences with at least one 1 and the first 1 occurring before the first 0 (possibly no 0's) **[5]**

b) How many arrangements are there of TAMELY with either T before A or A before M or M before E? [By before, we mean anywhere before, not just immediately before.] **[5]**

Q8) a) How many ways are there to assign seven different city cars denoted C_1, C_2, C_3, C_4, C_5, C_6, C_7 to seven officers, O_1, O_2, O_3, O_4, O_5, O_6, O_7 if O_1 will not drive C_1 or C_3 ; O_2 will not drive C_1 or C_5 ; O_4 will not drive C_3 or C_6. ; O_5 will not drive C_2 or C_7 ; O_7 will not drive C_4 ; O_3 and O_6 will drive all the cars. **[5]**

b) How many permutations of the 26 letters are there that contain none of the sequences MATH, RUNS, FROM, or JOE? **[5]**

Total No. of Questions : 8]

SEAT No. :

P1029

[Total No. of Pages : 2

[5428]-302
M.A./M.Sc. (Semester - III)
MATHEMATICS
MT-702 : Field Theory
(2013 Pattern) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instruction to the candidates:

1) *Attempt any five questions of the following.*
2) *Figures to the right indicate full marks.*

Q1) a) Prove that the element α is algebraic over F if and only if the simple extension $F(\alpha)/F$ is finite. **[5]**

b) Show that $p(x) = x^3-2x-2$ is irreducible polynomial over $\mathbb{Q}$. Also compute $(1+\theta+\theta^2)(1-\theta)$ in $\mathbb{Q}(\theta)$, where θ is a root of P(x). **[3]**

c) Find the fixed field of Aut $\left[\mathbb{Q}(\sqrt{2})/\mathbb{Q}\right]$ **[2]**

Q2) a) If F is a field and $p(x) \in F[x]$ is an irreducible polynomial then prove that there exists a field k containing an isomorphic copy of F in which p(x) has a root. **[5]**

b) Determine the degree of the extension $\mathbb{Q}\left(\sqrt{3+2\sqrt{2}}\right)$ over $\mathbb{Q}$. **[3]**

c) Find the smallest extension of $\mathbb{Q}$ having a root of $x^2-2 \in \mathbb{Q}[x]$. **[2]**

Q3) a) Find the splitting field for the polynomial **[5]**

$p(x) = x^3-2$ over $\mathbb{Q}$.

b) Show that trisecting an angle is impossible by using compass and straightedge. **[3]**

c) Is $\mathbb{Q}\left(\sqrt{2}\right)$ and $\mathbb{Q}\left(\sqrt{3}\right)$ isomorphic field? Justify. **[2]**

P.T.O.

Q4) a) Show that the Galois group of $(x^2-2)(x^2-3)$ is isomorphic to the klein 4-group. **[5]**

b) state fundamental theorem of Galois theory. **[3]**

c) Is the polynomial $(x^2-2)^n$ for any $n\geq 2$separable over $\mathbb{Q}$? **[2]**

Q5) a) If F, K and L are fields such that

$F \subseteq K \subseteq L$ then prove that $[L:F] = [L:K]\,[K:F]$. **[5]**

b) If F is a field and $\overline{F}$ is an algebraic closure of F then prove that F is algebraically closed. **[3]**

c) Define the following terms.

i) Normal extension of a field.

ii) Separable extension of a field **[2]**

Q6) a) Show that the finite field $\mathbb{F}_{p^n}$ is simple extension of $\mathbb{F}$. **[5]**

b) Show that every irreducible polynomial over a field of characteristic 0 is separable. **[3]**

c) Is the Galois group of the polynomial $x^3 - x+1 = 0$ solvable? **[2]**

Q7) a) Show that the cyclotomic polynomial $\Phi_n(x)$ in $\mathbb{Z}[x]$ is irreducible polynomial. **[5]**

b) Define the discriminant of a polynomial and obtain the discriminant of a cubic polynomial x^3+ax^2+bx+c. **[5]**

Q8) a) Let $\phi : F \to F'$ be an isomorphism of fields, $f(x) \in F[x]$ be a polynomial and $f'(x) \in F'[x]$ be the polynomial obtained by applying ϕ to the coefficients of $f(x)$. Let E be a spliting field for $f(x)$ over F and E' be a spliting field for $f'(x)$ over F'. then prove that the isomorphism ϕ extends to an isomorphism $\sigma : E \to E'$. **[5]**

b) True or false and Justify :

"A Galois extension of a Galois extension is Galois". **[5]**

Total No. of Questions : 8]

SEAT No. :

P1030

[Total No. of Pages : 2

[5428]-303
M.A./M.Sc. (Semester - III)
MATHEMATICS
MT-703 : Functional Analysis
(2013 Pattern) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) State and prove Cauchy-Bunyakowasky-Schwarz inequality. **[5]**

b) Define Hilbert Space. **[2]**

c) Give an example of a Hilbert space with detail explanation. **[3]**

Q2) a) If ε is an orthonormal set and h belongs to Hilbert space $\mathcal{H}$, then prove that $\Sigma\{\langle h, e\rangle\ e : e \in \varepsilon\}$ converges in $\mathcal{H}$. **[5]**

b) Define $V : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $(V f)(t) = f(t-1)$ and show that V is an isomorphism. **[3]**

c) Give an example of a (non-identity) unitary operator on an infinite dimensional Hilbert space. **[2]**

Q3) a) If $A \in \mathcal{B}(\mathcal{H})$ then prove that $\|A\| = \|A^*\| = \|A^*A\|^{1/2}$. **[4]**

b) If $A, B \in \mathcal{B}(\mathcal{H})$, then prove that $(AB)^* = B^*A^*$ and $A^{**} = (A^*)^* = A$. **[3]**

c) If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{K}, \mathcal{L})$, then prove that $BA \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ and $\|BA\| \le \|B\|\,\|A\|$. **[3]**

Q4) a) If T is a compact operator on a Hilbert space $\mathcal{H}$, $\lambda \neq 0$ and $\inf\{\|(T-\lambda)h\| : \|h\| = 1\} = 0$, then prove that $\lambda \in \sigma_p(T)$. **[4]**

b) Show that diagonalizable operator is normal. **[3]**

c) Give an example of an operator on a Hilbert space which has no eigenvalues. Justify your answer. **[3]**

P.T.O.

Q5) a) Prove that E is an idempotent operator if and only if $I - E$ is an idempotent operator. **[4]**

b) Show that for a separable Hilbert space $\mathcal{H}$ with a orthonormal basis $\{e_n\}$, if A is an operator defined by $Ae_n = \frac{1}{n}e_n$, then A is a compact operator. **[3]**

c) Give an example of a convex set in a Hilbert space. **[3]**

Q6) a) Let $\mathcal{X}$ be a finite dimensional normed space and let $\mathcal{Y}$ be and normed space. If $T : \mathcal{X} \to \mathcal{Y}$ is a linear transformation, then prove that T is continuous. **[4]**

b) Give an example of a Banach space $\mathcal{X}$ and a closed subspace $\mathcal{M}$ such that the natural map $Q : \mathcal{X} \to \mathcal{X}/\mathcal{M}$ is not closed. **[3]**

c) Show that l^∞ is not separable. **[3]**

Q7) a) State and prove the Open Mapping Theorem. **[5]**

b) i) Show that c^* is isometrically isomorphic to l^1. **[3]**

ii) If $\mathcal{X}$ is a normed space and $\mathcal{M}$ is a hyperplane in $\mathcal{X}$, then prove that either $\mathcal{M}$ is closed or $\mathcal{M}$ is dense. **[2]**

Q8) a) State and prove the Principle of Uniform Boundedness. **[5]**

b) i) Give an example of a Banach space which is not a Hilbert space. **[3]**

ii) State Hanh-Banach Theorem. **[2]**

Total No. of Questions : 8]

SEAT No. :

P1031

[Total No. of Pages : 2

[5428]-401

M.A./M.Sc. (Semester - IV)

MATHEMATICS

MT - 801 : Number Theory

(2013 Pattern) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) If m_1 and m_2 denote two positive relatively prime integers then prove that $\phi(m_1m_2) = \phi(m_1)\,\phi(m_2)$, where $\phi(m)$ is Euler's ϕ-function. **[5]**

b) If $ax \equiv ay \pmod m$ and $(a, m) = 1$ then prove that $x \equiv y \pmod m$ **[3]**

c) Is $x^2 \equiv 3 \pmod{13}$ solvable? Justify. **[2]**

Q2) a) State and prove the Gaussian reciprocity Law. **[5]**

b) Prove that product of four consecutive integers is divisible by 24. **[3]**

c) Show that $1^2, 2^2$, ----, m^2 is not a complete residue system modulo m if $m > 2$. **[2]**

Q3) a) Let $f(n)$ be a multiplicative function and let $F(n) = \sum_{d|n} f(d)$. Then prove that F(n) is multiplicative. **[5]**

b) What is the highest power of 7 dividing 1000! **[3]**

c) For any positive real numbers x and y prove that **[2]**

$[x - y] \leq [x] - [y] \leq [x - y] + 1$

Q4) a) If p ia a prime number and $p \equiv 1 \pmod 4$ then prove that there exist positive integers a and b such that $a^2 + b^2 = p$. **[5]**

b) Show that 1763 is composite by using Fermat's congruence. **[3]**

c) Find a positive integer n such that **[2]**

$\mu(n) + \mu(n + 1) + \mu(n + 2) = 3$

P.T.O.

Q5) a) Prove that the product of two primitive polynomials is primitive. **[5]**

b) Find all solutions of $10x-7y=17$. **[5]**

Q6) a) If α is any algebraic number then prove that there is a rational integer b such that b_α is an algebraic integer. **[5]**

b) Prove that if p is an odd prime then $x^2 \equiv 2 \pmod p$ has solutions if and only if $p \equiv 1$ or $7 \pmod 8$. **[5]**

Q7) a) If α and β are algebraic numbers then prove that $\alpha+\beta$ and $\alpha\beta$ are also algebraic numbers. **[5]**

b) Prove that $\left(\frac{P}{Q}\right)\left(\frac{P'}{Q}\right)=\left(\frac{PP'}{Q}\right)$, where $\left(\frac{P}{Q}\right)$ is the Legendre symbol. **[3]**

c) Prove that every integer is of the form 3k or of the form 3k+1 or of the form 3k+2. **[2]**

Q8) a) State and prove Möbius inversion formula. **[5]**

b) Let g be a primitive root of an odd prime p prove that the quadratic residue modulo p are congruent to g^2, g^4, g^6, ----, g^{p-1} and that the nonresidues are congruent to g, g^3, g^5, ---, g^{p-2}. **[3]**

c) Find $d(15)$, $\sigma(15)$, $\sigma_2(15)$, $\omega(15)$. **[2]**

Total No. of Questions : 8]

SEAT No. :

P1032

[Total No. of Pages : 2

[5428]-402
M.A./M.Sc. (Semester - IV)
MATHEMATICS
MT-802 : Differential Geometry
(2013 Pattern)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Determine whether the vector field $X(x_1, x_2) = (x_1, x_2, 1+ x_1^2, 0)$ is complete or not. **[4]**

b) Find and sketch the gradient field of the function $f(x_1, x_2) = x_1 + x_2$. **[4]**

c) Define the term : graph of a function. **[2]**

Q2) a) Explain why an integral curve cannot cross itself as does the parameterized curve. **[4]**

b) Show that gradient of f at $p \in f^{-1}(c)$ is orthogonal to all vectors tangent to $f^{-1}(c)$ at p. **[4]**

c) Define n-surface in $\mathbb{R}^{n+1}$ with example. **[2]**

Q3) a) State and prove Lagrange's Multiplier Theorem for n-surface. **[5]**

b) Show that the möbius band is an unorientable 2-surface. **[5]**

Q4) a) Show that the Ideingarten map is self - adjoint operator. **[4]**

b) Let $\alpha(t) = (x(t), y(t))$, $t \in I$ be local parameterization of the oriented plane curve C. Then show that **[4]**

$k o \alpha = (x'y'' - y'x'') / (x'^2 + y'^2)^{3/2}$

c) State Inverse function theorem. **[2]**

P.T.O.

Q5) a) Let $\alpha : I \to \mathbb{R}^{n+1}$ be parameterized curve with $\dot{\alpha}(t) \neq 0$ $\forall t \in I$. Show that there exists a unit speed reparameterization β of α. **[5]**

b) Find the global parameterization of the curve **[5]**

$(x_1 - a)^2 + (x_2 - b)^2 = \gamma^2$

Q6) a) Show that if S is a connected n-surface in $\mathbb{R}^{n+1}$ and $g : S \to \mathbb{R}$ is smooth and takes on only the +1 and –1 then g must be constant. **[5]**

b) Define the terms: **[2]**

i) Level sets

ii) Maximal integral curve

c) Let x be smooth vector field on an open set $U \subseteq \mathbb{R}^{n+1}$ and $P \in U$. Prove that there exists an open interval I containing zero and an integral curve $\alpha : I \to U$ of X such that $\alpha(0) = P$. **[3]**

Q7) a) Let $\phi : U \to \mathbb{R}^{n+1}$ be a parameterized n-surface in $\mathbb{R}^{n+1}$ and $P \in U$ then show that there exists an open set $U_1 \subset U$ about P such that $\phi(U_1)$ is an n-surface. **[5]**

b) Let x be smooth vector field along the parameterized curve $\alpha : I \to \mathbb{R}^{n+1}$ and f be smooth function along $\alpha(t)$. Then prove that $(f\dot{x}) = f\dot{x} + f'x$.**[3]**

c) Define curvature of a surface. **[2]**

Q8) a) Let S be an orientcd n-surface in $\mathbb{R}^{n+1}$ and $V \in S_p$ be unit vector, $p \in s$. Then show that there exist an open set $V \subseteq \mathbb{R}^{n+1}$ containing 'p' such that $S \cap \mathbb{N}(v) \cap V$ is a plane curve. **[5]**

b) Show that of X and Y are two parallel vector fields along $\alpha(t)$ then $X \cdot Y$ is constant along $\alpha(t)$. **[3]**

c) Define Covariant Differentiation. **[2]**

Total No. of Questions : 8]

SEAT No. :

P1033

[Total No. of Pages : 3

[5428]-403

M.A./M.Sc. (Semester - IV)

MATHEMATICS

MT - 803 : Fourier Analysis and Boundary Value Problems

(2013 Pattern) (Credit System)

Time : 3 Hours] *[Max. Marks : 50*

Instructions to the candidates:

1) ***Attempt any five questions.***

2) ***Figures to the right indicate full marks.***

Q1) a) Let f denote a function such that

i) f is continuous on the interval $-\pi \leq x \leq \pi$

ii) $f(-\pi) = f(\pi)$

iii) its derivatives f' is piecewise continuous on the interval $-\pi < x < \pi$. If a_n and b_n are the fourier coefficients $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)$ $\cos nx dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx$ for f, then prove that the series $\sum_{n=1}^{\infty}\sqrt{a_n^2 + b_n^2}$ converges. **[5]**

b) Find the fourier cosine series for the function $f(x) = \pi - x$ $(0 < x < \pi)$. **[3]**

c) Find the fourier sine series for the function $f(x) = x$ $(0 < x < \pi)$. **[2]**

Q2) a) If $f \in C_p(0, \pi)$, then prove that the Fourier sine series coefficient b_n tends to zero as n tends to infinity. **[5]**

b) Find the fourier series for the function **[3]**

$$f(x) = \begin{cases} 0 & \text{when} -\pi < x \leq 0 \\ x & \text{when}\ 0 < x < \pi. \end{cases}$$

P.T.O.

c) Let $f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right) & \text{when } x\neq 0 \\ 0 & \text{when } x=0. \end{cases}$

Then find f_R' (0) and f_L' (0) if it exist. **[2]**

Q3) a) Solve the following boundary value problem

$y_{tt}(x,t) = a^2 y_{xx}(x,t)$ $(0 < x < c, t > 0)$

$y(0, t) = 0, y(c, t) = 0, y_t(x, 0) = 0$

$y(x, 0) = f(x) \quad (0 \le x \le c).$ **[5]**

b) Solve the following boundary value problem

$u_t(x,t) = ku_{xx}(x,t)$ $(0 < x < \pi, t > 0)$

$u(0, t) = 0, u(\pi, t) = 0$ $(t > 0)$

$u(x, 0) = f(x) \quad (0 < x < \pi).$ **[5]**

Q4) a) Solve the following boundary value problem.

$u_{xx}(x, y) + u_{yy}(x, y) = 0$ $(0 < x < a,\ 0 < y < b)$

$u(0, y) = 0, u(a, y) = 0$ $(0 < y < b)$

$u(x, 0) = f(x), u(x, b) = 0$ $(0 < x < a).$ **[5]**

b) Solve the following boundary value problem

$u_t(x, t) = k\, u_{xx}(x, t) + q(t)$ $(0 < x < \pi, t > 0)$

$u(0, t) = 0, u(\pi, t) = 0$ and $u(x, 0) = f(x).$ **[5]**

Q5) a) Prove that necessary and sufficient condition for an orthonormal set $\{Q_n(x)\}$ $(n = 1, 2, 3, ...)$ to be complete is that for each function f in the space considered Parsevals equation $\sum_{n=1}^{\infty} c_n^2 = \|f\|^2$ where c_n are fourier constants $c_n = \langle f, \phi_n \rangle$ be satisfied. **[5]**

b) Derive the eigenvalue and eigenfunction of Sturm-Liouville problem $X''(x) + \lambda X(x) = 0,\ X'(0) = 0,\ X^1(c) = 0.$ **[3]**

c) Show that the function $\psi_1(x) = 1$ and $\psi_2(x) = x$ are orthogonal on interval $-1 < x < 1$ and determine constants A and B such that $\psi_3(x) = 1 + Ax + Bx^2$ is orthogonal to both $\psi_1(x)$ and $\psi_2(x)$ on the interval. **[2]**

Q6) a) If λ_m and λ_n are distinct eigenvalues of the Sturm-Liouville problem $[r(x)\, X'(x)]' + [q(x)+\lambda p(x)]\, X(x)=0$ $(a< x < b)$ under the conditions $a_1\, X(a) + a_2 X'(a) = 0$, $b_1 X(b) + b_2\, X'(b) = 0$, then prove that corresponding eigenfunctions $X_m(x)$ and $X_n(x)$ are orthogonal with respect to weight function $P(x)$ on the interval $a < x < b$. **[5]**

b) Find the eigenvalues and normalized eigenfunctions of Sturm - Liouville problem $X''(x) + \lambda X(x) = 0$, $X(0) = 0$, $X'(1) = 0$. **[3]**

c) Verify that each of the function $u_0 = y$ and $u_n = \sinh ny . \cos nx$ $(n = 1, 2, 3 \ldots\ldots)$ satisfies Laplace equation $u_{xx}(x,y) + u_{yy}(x,y) = 0$ $(0 < x < \pi,\ 0 < y < 2)$. **[2]**

Q7) a) Establish the recurrence relations

i) $$\frac{d}{dx}[x^{-n} J_n(x)] = -\, x^{-n} J_{n+1}(x) \quad (n = 0,1,2\ldots.)$$

ii) $$\frac{d}{dx}[x^{n} J_n(x)] = x^{n} J_{n-1}(x) \quad (n = 1, 2, 3\ldots.)$$ **[6]**

b) Derive Bessel's integral form

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x\sin\phi)\, d\phi \quad (n = 0, 1, 2\ldots.)$$ **[4]**

Q8) a) Solve the Legendre's differential equation $(1- x^2)\, y'' - 2xy' + \lambda y = 0$ **[5]**

b) Derive the Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n \quad (n = 0,1,2,\ldots\ldots)$$ **[3]**

c) If $F(x,t) = (1 - 2xt + t^2)^{\frac{-1}{2}}$, then show that $(1 - 2xt + t^2)\frac{\partial F}{\partial t} = (x - t)F$. **[2]**