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M.A./M.Sc.

MATHEMATICS

MT-501 : Real Analysis - I

(2008 Pattern) (Semester-I) (Theory)

Time : 3 Hours]   *[Max. Marks : 80*

Instructions to the candidates:

   1)   *Attempt <u>any five</u> questions.*

   2)   *Figures to the <u>right</u> indicate <u>full</u> marks.*

Q1) a)   If l^2 denotes the set of all square summable sequences of complex numbers then show that $\langle \{x_n\}, \{y_n\} \rangle = \sum_{n=1}^{\infty} x_n \overline{y}_n$ is a inner product on l^2.

                                                                                           [6]

  b)   Consider $\mathbb{R}$ with Euclidean metric. Give an open cover of (-10, 10] that has no finite subcover. **[5]**

  c)   Define interior of a set in a metric space (X, d) if A, B $\subset$ X then is it true that $\text{Int}(A \cup B) = \text{Int}(A) \cup \text{Int}(B)$? Prove or give a counter example. **[5]**

Q2) a)   Show that a subset of a metric space is compact iff it is sequentially compact. **[6]**

  b)   With usual notations, prove that m^* is countably additive on M. **[5]**

  c)   Let A = [0, 4] × (1, 10], B = (0,1] × [0,2] in $\mathbb{R}^2$. Draw the picture of the set S(A, B) and find D(A, B). **[5]**

Q3) a)   Define a measurable set in $\mathbb{R}^n$ and show that family of a measurable sets forms a σ-ring. **[6]**

  b)   Define a measurable functions and give it's other three equivalent definitions. **[5]**

  c)   Show that if f and g are two measurable functions then their product $f \cdot g$ is also measurable. **[5]**

Q4) a) Suppose A is a measurable set and $\{f_k\}$ is an increasing sequence of non-negative measurable functions and f_k converges to f pointwise then show that f is integrable and $\lim\limits_{k\to\infty}\int_A f_k dm = \int_A \lim\limits_{k\to\infty} f_k dm$. **[6]**

b) Prove that Cantor set is a Lebesgue measurable set and it has measure zero. **[5]**

c) Show that every continuous function is measurable. **[5]**

Q5) a) State and prove Lebesgue dominated convergence theorem. **[6]**

b) Give an example of a function f such that $|f|$ is measurable but f is not measurable. **[5]**

c) Give an example of a non-measurable function. **[5]**

Q6) a) With usual notations, for $1 \le p < \infty$ prove that $L^p(\mu)$ is complete. **[6]**

b) Show that step functions are dense in $L^p(\mu)$ for $1 \le p < \infty$. **[5]**

c) State Fatou's lemma and show that strict inequality holds for it. **[5]**

Q7) a) State and prove Holder's inequality. **[8]**

b) Show that classical Fourier series for $f(x) = x$ is $2\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n} \sin(nx)$. **[8]**

Q8) a) State and prove Banach contraction principle. **[8]**

b) Show that the series $\dfrac{1}{\sqrt{2\pi}}, \dfrac{\cos(nx)}{\sqrt{\pi}}, \dfrac{\sin(mx)}{\sqrt{\pi}}$ $n, m = 1, 2, 3 \dots$ is orthonormal in $L^2([-\pi,\pi],m)$. **[8]**

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[Total No. of Pages :3

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M.A. / M.Sc.

MATHEMATICS

MT- 502: Advanced Calculus

(Semester -I) (2008 Pattern)

Time : 3 Hours]   *[Max. Marks :80*

Instructions to the candidates:

    1)   ***Attempt any five questions:***

    2)   ***Figures to the right indicate full marks.***

Q1) a)   Let $\bar{f}, \bar{g}$ $S \to \mathbb{R}^m$ where $S \subset \mathbb{R}^n$ be vector fields and $\bar{a} \in \mathbb{R}^n$. Let

       $\lim\limits_{\bar{x} \to \bar{a}} \bar{f}(\bar{x}) = \bar{b}$ and $\lim\limits_{\bar{x} \to \bar{a}} \bar{g}(\bar{x}) = \bar{c}$ then prove that $\lim\limits_{\bar{x} \to \bar{a}} \left[\bar{f}(\bar{x}) \cdot \bar{g}(\bar{x}) \right] = \bar{b} \cdot \bar{c}$.

      [6]

     b)   Let $f(x, y) = \dfrac{x y^2}{x^2 + y^4}$ if $(x, y) \neq (0,0)$ and $f(0,0) = 0$. Show that all the

       directional derivatives exist at (0.0) but the function is not continuous at (0,0).   **[5]**

     c)   Evaluate the directional derivative of $f(x, y, z) = \left(\dfrac{x}{y} \right)^z$ at $(1, 1, 1)$ in the

       direction of $2\bar{i} + \bar{j} - \bar{k}$.   **[5]**

Q2) a)   Let $f: S \to \mathbb{R}$, $S \subset \mathbb{R}^n$ be a scalar field. Assume that the partial derivatives

       $D_1 f, \text{ --- } D_{nf}$ exist in some n-ball $B(\bar{a})$ and are continuous at $\bar{a}$, then

       prove that f is differentiable at $\bar{a}$.   **[8]**

     b)   If a scalar field f is differentiable at $\bar{a}$ then prove that f is continuous at

       $\bar{a}$.   **[4]**

     c)   Let z be a function of x and y where $x = u^2 + v^2 - 2uv$, $y = u + v$.

       Compute $(x + y) \dfrac{\partial z}{\partial x} + (x - y) \dfrac{\partial z}{\partial y}$.   **[4]**

P.T.O.

Q3) a) Define Line integral and illustrate it by an example. Also state the basic properties of Line integral. **[6]**

b) Let $\bar{f} = (f_1,....,f_n)$ be a continuously differentiable vector field on an open set S in $\mathbb{R}^n$. If $\bar{f}$ is gradient on S, then prove that the partial derivatives of the components of $\bar{f}$ are related by the equation $D_i f_j(\bar{x}) = D_j f_i(\bar{x})$ for $i, j = 1, 2,........., n$ and every $\bar{x} \in$ S. **[5]**

c) Evaluate the line integral of the vector field $\bar{f}(x, y\ z) = (y^2 - z^2)\ \bar{i} + 2yz\ \bar{j} - x^2\bar{k}$, along the path described by $\bar{\alpha}(t) = t\bar{i} + t^2\ \bar{j} + t^3\ \bar{k}, 0 \le t \le 1$. **[5]**

Q4) a) State and prove first fundamental theorem for line integrals . **[8]**

b) Calculate the work done by constant force with help of line integrals .**[4]**

c) Let $\bar{f}$ be a vector field continuous on an open connected set S in $\mathbb{R}^n$? If the Line integral of $\bar{f}$ is zero around every piecewise smooth closed path in S then prove that the line integral of $\bar{f}$ is independent of the path in S. **[4]**

Q5) a) Let $\bar{f}(x, y) = P(x, y)\ \bar{i} + Q(x, y)\ \bar{j}$ be a vector field that is continuously differentiable on an open simply connected set S in the plane. Prove that f is a gradient an S if and only if $\dfrac{\partial p}{\partial y} = \dfrac{\partial \theta}{\partial x}$ every where on S. **[8]**

b) Transform the given integrals to one or more iterated integrals in polar Co - ordinates $\int_0^1 \left[\int_0^1 f(x, y)\, dy\right] dx$. **[6]**

c) Determine the volume of an n-dimensional interval. **[2]**

Q6) a) State and prove Green's theorem for plane regions bounded by piecewise smooth Jordan curve. **[6]**

b) Make a sketch of the region of integration and evaluate $\iiint_S \sqrt{x^2 + y^2} \, dx \, dy \, dz$ where S is the solid formed by the upper nappe of the cone $z^2 = x^2 + y^2$ and the plane Z = 1. **[5]**

c) Evaluate the Line integral using the Green's theorem $\oint_c y^2 dx + x \, dy$ where

C is the square with vertices $(\pm 1, \pm 1)$. **[5]**

Q7) a) Define fundamental vector product. Find the fundamental vector product for the surface with explicit representation. What are the singular points of the surface with explicit representation. **[6]**

b) Define the surface integral and explain the terms involved in it. **[5]**

c) Compute the area of the region cut from the plane $x + y + z = a$ by the cylinder $x^2 + y^2 = a^2$. **[5]**

Q8) a) State and prove Gauss divergence theorem **[8]**

b) Determine the Jacobian matrix and compute the curl and divergence of the following vector field $\overline{F}(x, y, z) = (x^2 + yz)\overline{i} + (y^2 + xz)\overline{j} + (z^2 + xy)\overline{k}$. **[6]**

c) Define simple parametric surface. **[2]**

$$\mathcal{E} \quad \mathcal{E} \quad \mathcal{E}$$

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M.A./M.Sc.

MATHEMATICS

MT - 503 : Linear Algebra

(2008 Pattern) (Semester - I)

Time : 3 Hours]   *[Max. Marks : 80*

Instructions to the candidates:

  1)   Attempt any five questions.

  2)   Figures to the right indicate full marks.

Q1) a)   Let V and V' be finite dimensional vector spaces over K of dimensions n and m respectively. Prove that dim L(V, V') = nm.   **[6]**

   b)   Let V be finite dimensional vector space over K then prove that the following statements are equivalent for a subset B of V.   **[6]**

   i)   B is a basis

   ii)   B is a minimum generating set, that is no subset of B can generate V.

   iii) B is a maximal linearly independent set.

   c)   Let I = (−a, a), a > 0 be an open interval in R and let V = R^I, the space of all real valued functions defined on I. Show that $V = V_e \oplus V_0$, where V_e is the set of all even functions on I and V_0 is the set of all odd functions on I.   **[4]**

Q2) a)   Let $T: V \rightarrow V'$ be a Linear transformation. Prove that T is injective if and only if ker (T) = {0}.   **[6]**

   b)   Let V be a vector space over K and Let W_1 and W_2 be subspaces of V then show that $(W_1+W_2)/W_2 \cong W_1/W_1 \cap W_2$.   **[6]**

   c)   Consider the vector space $R_3[X]$ of polynomials with real coefficients and of degree at most 3. The differential operator D is a linear operator on $R_3[x]$. Write the matrix representation of D with respect to B_1 = {1 + x, x + x^2, x^2 + x^3, x + x^3}.   **[4]**

P.T.O.

Q3) a) Let $A \in K^{n \times n}$, the left multiplication by A defines a linear operator $\lambda_A : K^{n \times m} \to K^{n \times m}$ such that $\lambda_A(B) = AB$. Prove that α is an eigenvalue of λ_A if and only if α is an eigenvalue of A. **[6]**

b) let V and W be finite dimensional vector spaces Over K and let $T \in L(V,W)$. **[6]**

Prove that
i) $\ker T^* = (\text{im } T)^o$

ii) $\text{im } T^* = (\ker T)^o$

iii) $\text{rank}(T) = \text{rank}(T^*)$

c) If T is an invertible linear operator on a finite dimensional vector space over a field k, then prove that the minimal polynomial of T^{-1} is $M_T(0)^{-1}$ $X^r M_T\left(\dfrac{1}{X}\right)$, where $r = \deg M_T(X)$. **[4]**

Q4) a) State and prove the primary decomposition theorem. **[8]**

b) Prove that the geometric multiplicity of an eigenvalue of a linear operator can not exceed its algebraic multiplicity. **[6]**

c) Determine whether $T : R^3 \to R^3$

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+1 \\ y \\ z \end{bmatrix}$$ linear transformation. **[2]**

Q5) a) Let V be a finite dimensional vector space over K of dimension n and let T be a linear operator on V. If the characterisitics polynomial of T splits over K, then Prove that T is triangulable. **[8]**

b) Prove that a Jordan Chain consists of linearly independent vectors. **[4]**

c) The characteristics polynomial of a matrix is $(x-1)^3(x-2)^2$. Write its Jordan canonical forms. **[4]**

Q6) a) Give all possible rational canonical forms it the characteristics polynomial is : **[6]**

    i) $(x^2 + 2)(x - 3)^2$

    ii) $(x - 1)^2 (x + 1)^2$

    b) Let V be a finite dimensional vector space over K and let T be a linear operator on V. Prove that V is a direct sum of T cyclic subspaces. **[8]**

    c) Define : Jordan canonical form. **[2]**

Q7) a) Prove the polarization identities for the inner product space. **[4]**

    b) Let V be a finite dimentional inner product space and let f be a linear functional on V. Prove that there exists a unique vector x in v such that $f(v) = (v, x)$, for all v in V. **[8]**

    c) Prove that the eigen values of a unitary operator have absolute value 1. **[4]**

Q8) a) Let T be a self adjoint operator on an inner product space v. Prove that all roots of characteristic polynomials of T are real. **[5]**

    b) Consider the inner product space $R_3[x]$ with the inner product

$$\langle p(x), q(x) \rangle = \int_{-1}^{1} p(x) q(x) \, dx.$$ Find the adjoint of the differential

operator D. **[5]**

    c) Find a polar decomposition of the following matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. **[6]**

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M.A./M.Sc.

MATHEMATICS

MT-504 : Number Theory

(2008 Pattern) (Semester - I)

Time : 3 Hours]   *[Max. Marks :80*

Instructions to the candidates:
   1)   Attempt any five questions.
   2)   Figures to the right indicate full marks.

Q1) a)   Let p denote a prime. Then prove that $x^2 \equiv -1 \pmod{p}$ has solutions if and only if $p = 2$ or $p \equiv 1 \pmod 4$.   **[6]**

   b)   Show that the product of three consecutive integers is divisible by 504 if the middle one is cube.   **[5]**

   c)   Determine the value of $999^{179} \pmod{1763}$.   **[5]**

Q2) a)   If $f(n) = \sum_{d|n} \mu(d) F(n/d)$ for every positive integer n then prove that

   $F(n) = \sum_{d|n} f(d)$.   **[6]**

   b)   Find all primes p such that $x^2 \equiv 13 \pmod{p}$ has a solution.   **[5]**

   c)   Let x & y be any two real numbers then prove that   **[5]**

   i) $[x] + [y] \le [x+y] \le [x] + [y] + 1$

   ii) $[x] + [-x] = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise.} \end{cases}$

Q3) a) If p and q are distinct odd primes then prove that

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{\{(p-1)/2\}\{(q-1)/2\}}.$$ **[6]**

b) For every positive integer n, prove that $\delta(n)=\prod_{P^{\alpha}\|n}\left(\frac{P^{\alpha+1}-1}{P-1}\right)$. **[5]**

c) Prove that if a prime p is a quadratic residue of an odd prime q and p is of the form $4k+1$ then q is a quadratic residue of p. **[5]**

Q4) a) Show that : **[8]**

 i) The norm of product equals the product of norms $N(\alpha\beta) = N(\alpha).N(\beta)$.

 ii) $N(\alpha) = 0$ if and only if $\alpha = 0$.

 iii) The norm of an integer in $Q(\sqrt{m})$ is a rational integer.

b) What is the highest power of 2 dividing 533!. **[4]**

c) Find the minimal polynomial of $1+\sqrt{2}+\sqrt{3}$. **[4]**

Q5) a) Prove that every Euclidean quadratic field has the unique factorization property. **[8]**

b) If α is an algebraic integer then prove that there exists an integer b such that $b\alpha$ is an algebraic integer. **[4]**

c) Show that there is no x for which both $x\equiv 29\,(\text{mod }52)$ and $x\equiv 19\,(\text{mod }72)$. **[4]**

Q6) a) State and prove the Chinese remainder theorem. **[6]**

b) Find all integers x & y that satisfy $147x+258y=369$. **[5]**

c) Evaluate : $\left(\dfrac{-23}{83}\right)$. **[5]**

Q7) a) Let $Q(\sqrt{m})$ have the unique factorization property then prove that any rational prime p is either a prime π of the field or a product $\pi_1\pi_2$ of two primes, not necessarily distinct of $Q(\sqrt{m})$. **[6]**

b) Prove that $\sum_{j=1}^{p-1}\left(\dfrac{j}{p}\right)=0$, P an odd prime. **[5]**

c) What is the last digit in the ordinary decimal representation of 2^{400}? **[5]**

Q8) a) Let a, b and c be integer with not both a and b equal to 0 and let $g = $ $gcd(a, b)$ and $ax + by = c$.
  i)   If $g \nmid c$ then show that the equation $ax + by = c$ has no solution in integer.
  ii) If $g|c$ then prove that $ax + by = c$ has infinitely many solutions. **[6]**

b) Show that $61!+1\equiv 63!+1\equiv 0\,(\text{mod } 71)$. **[5]**

c) Prove that $\prod_{d|n} d = n^{d(n)/2}$. **[5]**

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M.A/M.Sc.

MATHEMATICS

MT - 505 : Ordinary Differential Equations

(2008 Pattern) (Semester - I)

Time : 3 Hours]   *[Max. Marks : 80*

Instructions to the candidates:

 1) *Attempt any five questions.*

 2) *Figures to the right indicate full marks.*

Q1) a)   Find orthogonal trajectories to the family of curves $y^2 = 4c(x+c)$. Also sketch these families.   **[7]**

   b)   Find the differential equation of the following one-parameter family of curves: $y = x\sin(x+c)$.   **[6]**

   c)   Verify that one solution of $xy'' + (2x+1)y' + (x+1)y = 0$ is given by $y_1 = e^x$.   **[3]**

Q2) a)   If $y_1(x)$ and $y_2(x)$ are two solutions of equation $y'' + P(x)y' + Q(x)y = 0$ on [a,b], then their wronskian $W = W(y_1 y_2)$ is either identically zero or never zero on [a, b].   **[6]**

   b)   Find the general solution of $y'' - xf(x)y' + f(x)y = 0$.   **[5]**

   c)   Solve the following initial value problem: $y'' - 5y' + 6y = 0$ $y(1) = e^2$ and $y'(1) = 3e^2$ **[5]**

Q3) a) If $y_1(x)$ is one solution of the differential equation $y'' + P(x)y' + Q(x)y = 0$ then find the other solution y_2. **[5]**

b) Find the general solution of the following Euler equation:
$2x^2 y'' + 10xy' + 8y = 0$ **[6]**

c) Show that zeros of the functions $a \sin x + b \cos x$ and $c \sin x + d \cos x$ are distinct and occur alternately whenever $ad - bc \neq 0$ **[5]**

Q4) a) State and prove sturm comparison theorem. **[5]**

b) Find the general solution of $(1 + x^2)y'' + 2xy' - 2y = 0$ in terms of power series in x. **[7]**

c) Classify the singular points of the differential equation
$x^3(x-1)y'' - 2(x-1)y' + 3xy = 0$ **[4]**

Q5) a) State and solve second order Legendre ordinary differential equation.**[8]**

b) Find two independent Frobenius series solutions of the equation
$x^2 y'' - xy' + (x^2 - 2)y = 0$. **[8]**

Q6) a) Find the general solution of the system: $\begin{cases} \dfrac{dx}{dt} = 7x + 6y \\ \dfrac{dy}{dt} = 2x + 6y \end{cases}$. Also find wronskian of its two independent solutions. **[8]**

b) Obtain volterra prey - predator differential equations about fox - rabbit population problem. Also explain method of linearization. **[8]**

Q7) a) For the system:
$$\begin{cases} \dfrac{dx}{dt} = y\left(x^2 + 1\right) \\ \dfrac{dy}{dt} = 2xy^2 \end{cases}$$

     i)   Find the critical points.

     ii)   Find the differential equation of the paths.

     iii)   Solve the eq^n obtained in (ii)   **[8]**

   b)   Determine the nature and stability properties of the critical point (0, 0)

for the following linear autonomous system:
$$\begin{cases} \dfrac{dx}{dt} = -x - 2y \\ \dfrac{dy}{dt} = 4x - 5y \end{cases}$$
                [8]

Q8) a) Find the exact solution of the initial value problem $y' = 2x(1 + y),\ y(0) = 0$.

Also calculate $y_1(x), y_2(x), y_3(x), y_4(x)$ using Picards method with

$y_0(x) = 0$.   **[8]**

   b)   Solve the following initial value problem by Picards method and compare

the result with exact solution:
$$\begin{cases} \dfrac{dy}{dx} = z \quad y(0) = 1 \\ \dfrac{dz}{dx} = -y \quad z(0) = 0 \end{cases}$$
          [8]

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M.A./M.Sc.

MATHEMATICS

MT - 601 : General Topology

(2008 Pattern) (Old course) (Semester - II)

Time : 3 Hours]   *[Max. Marks : 80*

Instructions to the candidates:
  1) *Attempt any five questions.*
  2) *Figures to the right indicate full marks.*

Q1) a) Let $\mathbb{B}$ be a collection of non-empty sets (not necessarily disjoint). Prove that the exists a function

  $$c : \mathbb{B} \to \bigcup_{B \in \mathbb{B}} B$$

  Such that c(B) is an element of B, for each $B \in \mathbb{B}$. **[8]**

  b) Prove that the topology $\mathbb{R}_l$ on $\mathbb{R}$ is strictly finer than the standard topology on $\mathbb{R}$. **[4]**

  c) Suppose X and Y are two topological spaces. Show that the collection.

  $$S = \left\{ \pi_1^{-1}(\cup) \mid \cup \text{ is open in } X \right\} \cup \left\{ \pi_2^{-1}(V) \mid V \text{ is open in } Y \right\}$$

  forms a sub basis for product topology on X×Y. **[4]**

Q2) a) If $\{\tau_\alpha\}$ is a family of topologies on X, show that $\bigcap \tau_\alpha$ is a topology on X. Is $\bigcup \tau_\alpha$ a topology on X. Justify. **[6]**

  b) Prove that the order topology on $\mathbb{Z}_+$ is the discrete topology on $\mathbb{Z}_+$. **[6]**

  c) Let Y = [–1,1],which of the following sets are open in Y? Which are open in $\mathbb{R}$? Justify.

i) $A = \left\{ x \mid \frac{1}{2} \leq \mid x \mid \leq 1 \right\}$

ii) $B = \left\{ x \mid \frac{1}{2} < \mid x \mid \leq 1 \right\}$. **[4]**

Q3) a) Let A be a subset of the topological space X. Prove that $x \in \overline{A}$ if and only if every open set U containing x intersects A. **[6]**

b) Let X be a space satisfying T_1 axiom and A be a subset of X. Prove that the point x is a limit point of A if and only if every neighbourhood of x contains lnfinitely many points of A. **[6]**

c) Show that every simply ordered set is a Hausdorff space in the order topology. **[4]**

Q4) a) Prove that every finite point set in a Hausdorff space X is closed. **[6]**

b) Show that the topological space X is Hausdorff if and only if the diagonal $\Delta = \{x \times x \mid x \in X\}$ is closed in X×X. **[6]**

c) State and prove the pasting lemma. **[4]**

Q5) a) Let A be any set, X and Y be two topological spaces with $f : A \rightarrow X \times Y$ defined by $f(a) = (f_1(a), f_2(a))$. Then show that f is continuous if and only if f_1 and f_2 are continuous, where $f_1 : A \rightarrow X$ and $f_2 : A \rightarrow Y$. **[8]**

b) Prove that the image of a connected space under a continuous map is connected. **[4]**

c) Let $\{A_n\}$ be a sequence of connected subspaces of X such that $A_n \cap A_{n+1} \neq \phi$, for all n. Show that $\bigcup_n A_n$ is connected. **[4]**

Q6) a) Prove that a topological space X is locally connected if and only if for every open set $\cup$ of X, each component of $\cup$ is open in X. **[6]**

b) Show that the compact subspace of a Hausdorff is closed. **[6]**

c) Define quotient topology and give an example of a quotient map which is not an open map. **[4]**

Q7) a) State and prove the tube lemma. **[8]**

b) Prove that compactness implies limit point compactness. Is converse true? Justify. **[5]**

c) Show that not every first countable space is second countable. **[3]**

Q8) a) State and prove the Tychonoff theorem. **[12]**

b) State : **[4]**

    i) Tietze extension theorem.

    ii) The Urysohn lemma.

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M.Sc. / M.A.

MATHEMATICS

MT - 602 : Differential Geometry

(2008 Pattern) (Semester - II)

Time : 3 Hours]   *[Max. Marks :80*

Instructions to the candidates:

   1)   ***Attempt any five questions.***

   2)   ***Figures to the right indicate full marks.***

Q1) a)   Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \rightarrow S$ be a parametrized curve in S, let $t_0 \in I$ and $V \in S_{\alpha(t_0)}$. Prove that there exists a unique vector field V tangent to S along α and $V(t_0) = V$. **[6]**

   b)   Find the velocity, acceleration and speed of the curve $\alpha(t) = (cost, sint)$. **[4]**

   c)   Consider a vector field $X(x_1, x_2) = (x_1, x_2, 1, 0)$ on $\mathbb{R}^2$. For $t \in \mathbb{R}$ and $p \in \mathbb{R}^2$, let $\phi_t(p) = \alpha_p(t)$ where α is the maximal integral curve of X through P. Show that $F(t) = \phi_t$ is a homomorphism of additive group of real numbers into the invertible linear maps of the plane. **[6]**

Q2) a)   Let S be a connected n - surface in $\mathbb{R}^{n+1}$. Show that on S there exists exactly two smooth unit normal vector fields N_1 and N_2. **[6]**

   b)   Show that the Weingarten map of the n-sphere of radius r oriented by inward normal is multiplication by $\frac{1}{r}$. **[5]**

   c)   Show that the 1-form n on $\mathbb{R}^2$-{0} defined by

$$n = \frac{-x_2}{x_1^2 + x_2^2} dx_1 + \frac{x_1}{x_1^2 + x_2^2} dx_2 \text{ is not exact.}$$ **[5]**

P.T.O.

Q3) a) Show that the covariant differentiation has the following property:

$(X.Y)' = X'.Y + X.Y'$ **[5]**

b) Let $a,b,c,d \in \mathbb{R}$ be such that $ac-b^2 > 0$. Show that the maximum and minimum values of the function $g(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$ on the unit circle $x_1^2 + x_2^2 = 1$ are eigen values of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. **[6]**

c) If an n-surface S contains a straight line segment, then show that it is geodesic in S. **[5]**

Q4) a) Let U be an open subset of $\mathbb{R}^{n+1}$ and $f : U \to \mathbb{R}$ be a smooth function. Let $S = f^{-1}(c)$ $c \in \mathbb{R}$ and $\nabla f(q) \neq 0 \; \forall q \in S$. If $g : U \to \mathbb{R}$ is smooth function and $P \in S$ is an extreme point of g on S, then show that $\exists$ a real number λ such that $\nabla g(p) = \lambda \nabla f(p)$. **[6]**

b) Let S denote a cylinder $x_1^2 + x_2^2 = r^2$ of radius r in $\mathbb{R}^3$. Show that α is a geodesic of S if and only if α is of the form $\alpha(t) = (r\cos(at + b), r\sin(at + b), ct + d)$ for some real numbers a,b,c,d. **[6]**

c) Sketch the following vector fields on $\mathbb{R}^2 : X(P) = (P, X(P))$ where

   i) $X(p) = -p$

   ii) $X(x_1, x_2) = (-x_2, x_1)$. **[4]**

Q5) a) Show that the Weingarten map Lp is self-adjoint (that is

$L_p(v).w = v.L_p(w) \; \forall v, w \in S_p$.) **[6]**

b) Find the integral curve of the vector field X given by

$X(x_1,x_2) = (x_1,x_2\text{-}x_2,x_1)$ through the point (1, 0). **[5]**

c) Let $\alpha(t) = (x(t), y(t))$ be a local parametrization of the oriented plane curve C. Show that $K o\alpha = \dfrac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$. **[5]**

Q6) a) Prove that on each compact oriented n-surface S in $\mathbb{R}^{n+1}$ there exists a point p such that the second fundamental form at p is definite. **[6]**

b) Show that speed of geodesic is constant. **[5]**

c) Show that the graph of any smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is an n-surface in $\mathbb{R}^{n+1}$ **[5]**

Q7) a) Let U be an open subset of $\mathbb{R}^{n+1}$ and $f : U \to \mathbb{R}$ be a smooth function. Let $p \in U$ be a regular point of f and let $c = f(p)$. Show that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $\left[\nabla f(p)\right]^{\perp}$. **[6]**

b) Let C be connected oriented plane curve and $\beta : I \to C$ be a unit speed global parametrization of C. Show that β is either one to one or periodic. **[5]**

c) Find the curvature of the circle with center (a, b) and radius r oriented by outward normal. **[5]**

Q8) a) Let S be an n-surface in $\mathbb{R}^{n+1}$ and $p \in S$. Prove that there exists an open set V about p in $\mathbb{R}^{n+1}$ and a parametrized n-surface $\phi : U \to \mathbb{R}^{n+1}$ such that ϕ is one to one map from U onto $V \cap S$. **[8]**

b) Let S be the ellipsoid $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{b^2} + \dfrac{x_3^2}{c^2} = 1$ a,b,c all non-zero, oriented by the outward normal. Show that the Gaussian curvature of S is

$$K(p) = \frac{1}{a^2 b^2 c^2 \left(\dfrac{x_1^2}{a^4} + \dfrac{x_2^2}{b^4} + \dfrac{x_3^2}{c^4} \right)}.$$ **[8]**

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[5221]-23

M.A./M.Sc.

MATHEMATICS

MT-603 : Groups and Rings

(2008 Pattern) (Semester-II) (New)

Time : 3 Hours]   *[Max. Marks : 80*

Instructions to the candidates:

   1)   *Attempt any five questions.*

   2)   *Figures to the right indicate full marks.*

Q1) a)   Define normal subgroup. Give an example of a group which is non abelian but has a proper normal subgroup. Justify the answer. **[5]**

    b)   If G be a finite group and $a, b \in G$, then prove that order(ab) = order(ba). **[5]**

    c)   Give an example of an infinite group whose every subgroup is of finite order. Justify the answer. **[6]**

Q2) a)   Prove that $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ are not isomorphic. Further, prove that $(\mathbb{R}, +)$ and $(\mathbb{Q}, +)$ are also not isomorphic. **[5]**

    b)   List all 6 cyclic subgroups of in U(15). **[5]**

    c)   Find Aut($\mathbb{Z}_8$), the group of automorphisms of $\mathbb{Z}_8$. **[6]**

Q3) a)   Let G be a group and Z(G) be the center of G. Prove that, if G/Z(G) is cyclic, then G is abelian. **[5]**

    b)   Let G be a group of functions from $\mathbb{R}$ to $\mathbb{R}^*$ under multiplication. Let $H = \{ f \in G \mid f(1) = 1 \}$. Prove that H is a subgroup of G. **[5]**

    c)   State and prove the Cayley's theorem. **[6]**

Q4) a) Find the inverse and the order of each of the following permutations in S_{19}
i) (1 4 9) (17 3 19 7) (2 6 5)
ii) (13 15 14) (11 12 6) (5 7 2). **[5]**

b) Show that every element in A_n for $n \geq 3$ can be expressed as a 3-cycle or a product of 3 cycles. **[5]**

c) State and prove the Lagrange's theorem for finite groups. Is the converse of the theorem true? Justify. **[6]**

Q5) a) State and prove the Orbit-Stabilizer theorem. **[5]**

b) Let G be a finite abelian group and let p be a prime that divides the order of G. Then prove that G has an element of order p. **[5]**

c) If $\tau = (5\ 7\ 3\ 4)(11\ 2), \rho = (12\ 4\ 2\ 9\ 1)(10\ 7\ 11) \in S_{12}$. Then find $\tau^{-1}\rho\tau$ and $\rho^{-1}\tau\rho$. **[6]**

Q6) a) Let G and H be finite abelian groups. Then prove that $G \oplus H$ is cyclic if and only if |G| and |H| are relatively prime. **[5]**

b) Determine all the homomorphisms from $\mathbb{Z}_{10}$ to $\mathbb{Z}_{25}$. **[5]**

c) Find all the non isomorphic abelian groups of order 6300. **[6]**

Q7) a) If H is a normal subgroup of a group G and N is a normal subgroup of H. Then is it true that N is a normal subgroup of G? Justify your answer. **[5]**

b) Let |G| = 2p, where p is an odd prime. Then prove that G is isomorphic to $\mathbb{Z}_{2p}$ or D_p, where D_p is dihedral group of order 2p. **[5]**

c) If H is a subgroup of a finite group G and |H| is a power of a prime p, then prove that H is contained in some Sylow p-subgroup of G. **[6]**

Q8) a) Let N be a normal subgroup of G, and |G/N| = m. Then show that $x^m \in N$ for all $x \in G$. **[5]**

b) Prove that every group of prime power order has non trivial center. **[5]**

c) Prove that the group of order 400 is not simple. **[6]**

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M.A./M.Sc.

MATHEMATICS

MT-604 : Complex Analysis

(2008 Pattern) (Semester-II)

Time : 3 Hours]   *[Max. Marks : 80*

Instructions to the candidates:

  1)   Attempt any five questions.

  2)   Figures to the right indicate full marks.

Q1) a)   Prove that the radii of convergence of $\sum\limits_{n=0}^{\infty} a_n z^n$ and $\sum\limits_{n=1}^{\infty} n a_n z^{n-1}$ are same.**[6]**

   b)   Let G be an open connected subset of $\mathbb{C}$ and let $f : G \to \mathbb{C}$ be differential such that $f'(z) = 0$ for all z in G. Prove that f is constant.   **[5]**

   c)   Find the radius of convergence of each of the series $\sum\limits_{n=1}^{\infty} 2^n z^n$ and $\sum\limits_{n=1}^{\infty} z^{n!}$.**[5]**

Q2) a)   Let G be an open disc in $\mathbb{C}$ and let $u : G \to \mathbb{R}$ be a harmonic function. Prove that u has a harmonic conjugate.   **[6]**

   b)   Show that a real-valued analytic function is constant.   **[5]**

   c)   Show that a Mobius transformation is a combination of translations, dilations and the inversion.   **[5]**

Q3) a)   Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}_{\infty}$. Prove that the cross ratio (z_1, z_2, z_3, z_4) is a real number if and only if all four points lie on a circle.

   [6]

   b)   Let G be a region and suppose $f : G \to \mathbb{C}$ is analytic such that $f(G)$ is contained in a subset of a circle. Show that f is constant.   **[5]**

   c)   Prove that a Mobius transformation has ∞ as its only fixed point if and only if it is a translation.   **[5]**

P.T.O.

Q4) a) Let $f : G \to \mathbb{C}$ be analytic and suppose the closure of the disc B$(a; r)$ is a subset of G. Prove that if $\gamma(t) = a + re^{it}, 0 \le t \le 2\pi$, then **[6]**

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw \text{ for } |z - a| < r.$$

b) Let f be analytic in the disc B$(a; R)$ and suppose that γ is a closed rectifiable curve in B$(a; r)$. Prove that $\int_\gamma f = 0$. **[5]**

c) Evaluate the integrals: **[5]**

i) $\int_{|z|=1} \frac{e^{iz}}{z^2} dz$

ii) $\int_{|z|=2} \frac{1}{z^2 + 1} dz$

Q5) a) State and prove Liouville's theorem. Hence deduce that $\sin z$ is not bounded. **[6]**

b) State and prove the maximum modulus theorem. **[5]**

c) Let f and g be two entire functions such that $fg \equiv 0$. Prove that $f \equiv 0$ or $g \equiv 0$. **[5]**

Q6) a) Let G be a simply connected region and $f : G \to \mathbb{C}$ be analytic in G. Prove that f has a primitive. **[6]**

b) Let $\gamma(\theta) = \theta e^{i\theta}$ for $0 \le \theta \le 2\pi$ and $\gamma(\theta) = 4\pi - \theta$ for $2\pi \le \theta \le 4\pi$.

Evaluate $\int_\gamma \frac{dz}{z^2 + \pi^2}$. **[5]**

c) Show that if $f : G \to \mathbb{C}$ is analytic and one-one, then $f'(z) \ne 0$ for any $z \in$ G. **[5]**

Q7) a) State and prove Goursat's theorem. **[6]**

b) Let G be a bounded region and suppose f is continuous on $\overline{G}$ and analytic on G. Show that if $|f(z)| = 2$ for all z on the boundary of G, then either f is a constant function or f has a zero in G. **[5]**

c) State and prove Schwarz's lemma. **[5]**

Q8) a) Suppose f has an essential singularity at $z = a$. Prove that for every $\delta > 0$, the set $f[ann(a; 0, \delta)]$ is dense in $\mathbb{C}$. **[6]**

b) Classify the singularities of $f(z) = \dfrac{z^2 - 1}{z(z-1)^2}$. Also, find the residue at each singularity. **[5]**

c) Using residue calculus, show that $\displaystyle\int_0^\pi \dfrac{d\theta}{a + \cos\theta} = \dfrac{\pi}{\sqrt{a^2 - 1}}$. **[5]**

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M.A./M.Sc.
MATHEMATICS
MT- 605: Partial Differential Equations
(2008 Pattern) (Old) (Semester - II)

Time : 3 Hours]   *[Max. Marks :80*

Instructions to the candidates:
1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*

Q1) a) Eliminate the parameters '*a*' and '*b*' from the equation $2Z = (ax + y)^2 + b$. **[5]**

b) Explain the method of solving the following first order partial differential equations. **[6]**

   i) $f(z, p, q) = 0$

   ii) $g(x, p) = h(y, q)$

c) Find the general integral of: $z(xp - yq) = y^2 - x^2$. **[5]**

Q2) a) If $h_1 = 0$ and $h_2 = 0$ are compatible with $f = 0$, then prove that h_1 and h_2 satisfy: **[5]**

$$\frac{\partial(f,h)}{\partial(x,u_x)} + \frac{\partial(f,h)}{\partial(y,u_y)} + \frac{\partial(f,h)}{\partial(z,u_z)} = 0 \text{ where } h = h_i, i = 1, 2.$$

b) Verify that the equation is integrable and find its solution

$$yz(y + z)dx + xz(x + z)dy + xy(x + y)dz = 0$$ **[6]**

c) Find the complete integral of the partial differential equation: **[5]**

$$(1 + yz)dx + z(z - x)dy - (1 + xy)dz = 0$$

Q3) a) Show that the equations: $f = xp - yq - x = 0, \; g = x^2 p + q - xz = 0$ are compatible and find a one parameter family of common solution. **[6]**

b) Find the complete integral of the equation: $p^2 x + q^2 y = z$, by Jacobi's method. **[5]**

c) Find the complete integral of the equation: $px + qy = pq$ by Charpit's method. **[5]**

Q4) a) Find the solution of the equation: **[8]**

$$Z = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y) \text{ which passes through x-axis.}$$

b) Reduce the equation: **[5]**

$$y^2 u_{xx} - 2xy\, u_{xy} + x^2 u_{yy} = \frac{y^2}{x} u_x + \frac{x^2}{y} u_y \text{ to canonical form and solve it.}$$

c) Derive the analytic expression for Monge cone at (x_0, y_0, z_0). **[3]**

Q5) a) Find the integral surface of the equation: $pq = z$, passing through $C : x_0 = 0, \; y_0 = s, \; z_0 = s^2$. **[5]**

b) Solve by Jacobi's method: $z^2 + 2U_z - U_x^2 - U_y^2 = 0$. **[6]**

c) State and prove Harnack's theorem. **[5]**

Q6) a) Prove that the solution of Neumann problem is unique upto addition of a constant. **[8]**

b) Find the solution of Heat-equation in finite rod, which is defined as: **[8]**

$$u_t = x\,u_{xx}, 0 < x < e, t > 0$$
$$u(0,t) = u(e,t) = 0, t > 0$$
$$u(x,0) = f(x), \; 0 \le x \le e.$$

Q7) a)   State Dirichilet's problem for a circle and find it's solution.   **[8]**

   b)   State and prove Kelvin's Inversion theorem.   **[8]**

Q8) a)   Using Duhamel's principle find the solution non-homogenous wave-equation: **[8]**

$$u_{tt} = x^2 \, u_{xx}, \, F(x,t), -\infty < x < \infty, t > 0$$
$$u(x,0) = u_t(x,0) = 0, -\infty < x < \infty$$

   b)   Classify the following equation in to Hyperbolic, Parabolic or Elliptic type.   **[8]**

   i) $7u_{xx} - 10\,u_{xy} - 22\,u_{yz} + 7\,u_{yy} - 16\,u_{zz} - 5\,u_{zz} = 0$

   ii) $e^z u_{xy} - u_{xx} = \log(x^2 + y^2 + z^2 + 1)$

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M.A./M.Sc.

MATHEMATICS

MT-606: Object Oriented Programming Using C++

(2008 Pattern) (Semester - II)

Time : 2 Hours]   *[Max. Marks :50*

Instructions to the candidates:

    1)   *Question 1 is compulsory.*

    2)   *Attempt any two from questions 2, 3, 4.*

    3)   *Figures to the right indicate full marks.*

Q1) Attempt the following questions.   **[20]**

    a)   Write a short note on function prototype.

    b)   What is data encapsulation?

    c)   What is use of scope resolution operator?

    d)   Write a function to read a matrix of size m×n from the keyboard using 'for' loop.

    e)   Write a function to find GCD of two numbers.

    f)   Give example of union in C++.

    g)   Write a note on operator "New".

    h)   Write a note on function overloading.

    i)   What is difference between private and protected members.

    j)   Which operators we can not overload?

Q2) a) Define a class rational having two data members numerator and denominator. Overload necessary constructors. And overload operators + and –. Find addition and substraction of two rational numbers. **[10]**

 b) Illustrate by example the use of virtual functions. **[5]**

Q3) a) Write a note on compile time polymorphism and run time polymorphism. **[10]**

 b) Write a note on inline functions. **[5]**

Q4) a) Illustrate by example, use of static member functions. **[10]**

 b) Write a note on const. member function. **[5]**

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[5221]-31

M.A./M.Sc.

MATHEMATICS

MT-701 : Functional Analysis

(2008 Pattern) (Semester-III)

Time : 3 Hours]   **[Max. Marks : 80**

Instructions to the candidates:

1) *Attempt any <u>five</u> questions.*
2) *Figures to the right indicate full marks.*

Q1) a) If M is a linear subspace of a Hilbert space H, show that M is closed if and only if $M^{\perp\perp} = M$. **[6]**

     b) Prove that a Hilbert space H is separable if and only if every orthonormal set in H is countable. **[6]**

     c) Prove that l_2 is a Hilbert Space. **[4]**

Q2) a) State and prove Bessel's Inequality. **[8]**

     b) Prove that convex subset C of a Hilbert space H contains a unique vector of smallest norm. **[6]**

     c) State Uniform Boundedness Theorem. **[2]**

Q3) a) If T is an operator on a Hilbert space H with $(Tx, x) = 0$ for all $x \in H$ then prove that T = 0. **[6]**

     b) Prove that an operator T on a Hilbert space H is self-adjoint if and only if (Tx, x) is real for all $x \in H$. **[6]**

     c) Give an example of linear map between Hilbert spaces, which is an isometry but not surjective. Justify your answer. **[4]**

Q4) a) Prove that an operator T on a Hilbert space H is normal if and only if $\|T^*x\| = \|Tx\|$ for every $x \in H$. **[5]**

     b) If a Hilbert space H is finite dimensional, show that every isometric isomorphism of H onto itself is unitary. **[5]**

     c) If P and Q are the projections on closed linear subspaces M and N of a Hilbert space H, prove that PQ is projection if and only if PQ = QP. **[6]**

P.T.O.

Q5) a) Show that the unitary operators on a Hilbert space H form a group. **[4]**

b) If T is a normal operator on a Hilbert space H, then prove that x is an eigenvector of T with eigenvalue λ if and only if x is an eigenvector of T^* with eigenvalue $\bar{\lambda}$. **[6]**

c) Show that an operator T on a Hilbert space H is normal if and only if its adjoint T^* is a polynomial. **[6]**

Q6) a) Let M be a closed linear subspace of a normed linear space N. If the norm of a coset $x + M$ in quotient space N/M is defined by

$$\|x + M\| = \inf \{\|x + m\| : m \in M\},$$ then prove that N/M is a normed linear space. Further, prove that if N is a Banach space, then so is N/M. **[6]**

b) Let N be a non-zero normed linear space, and prove that N is a Banach space if and only if $\{x : \|x\| = 1\}$ is complete. **[5]**

c) State and prove Holder's inequality. **[5]**

Q7) a) State and prove the Open Mapping Theorem. **[6]**

b) Let T be an operator on a banach space B. Show that T has an inverse T^{-1} if and only if T^* has an inverse $(T^*)^{-1}$, and that in this case $(T^*)^{-1} = (T^{-1})^*$. **[6]**

c) Let N and N' be normed linear spaces and T a linear transformation of N into N'. Prove that T is continuous if and only if T is continuous at origin. **[4]**

Q8) a) If X is a finite dimensional vector space over $\mathbb{F}$, then prove that any two norms on X are equivalent. **[6]**

b) Let X be a compact Hausdorff space, and justify the assertion that C(X) is reflexive if X is finite. **[4]**

c) State and prove Hanh-Banach Theorem. **[6]**

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[5221] - 32

M.A. / M.Sc.

MATHEMATICS

MT - 702 : Ring Theory

(2008 Pattern) (Old Course) (Semester - III)

Time : 3 Hours]   *[Max. Marks :80*

Instructions to the candidates:

   1)   *Attempt any five questions.*

   2)   *Figures to right indicate full marks.*

Q1) a)   Let $R = C(X, \mathbb{R})$ be the ring of all continuous functions on metric space X. Show that R is commutative ring with unity under addition and multiplication of maps. **[6]**

    b)   Show that $R = C(X, \mathbb{R})$ has no non-trivial idempotents (i.e idempotents other than 0 or 1) if and only if X is connected. **[6]**

    c)   Is $R = C(X, \mathbb{R})$ integral domain? Justify. **[4]**

Q2) a)   Let K be a field. Show that K[x] is Euclidean domain Is K[X, Y] Euclidean domain? Justify your answer. **[8]**

    b)   Show that every Euclidean domain is P.I.D. Give two examples of P.I.D's. **[8]**

Q3) a)   Show that there exist a maximal ideal in a ring with multiplicative identity. **[8]**

    b)   Show that every maximal ideal is prime. Is the converse true? Justify your answer. **[8]**

Q4) a) Let R be a commutative ring with unity. Show that R is integral domain if and only if R[X] is integral domain. **[8]**

b) Is $\mathbb{Z}[X]$ PID? **[4]**

c) Is $\mathbb{R}[X]/(x^3 + x + 1)$ PID? **[4]**

Q5) a) Show that every PID is unique factorization domain. **[6]**

b) State Eisenstein's criterion. Show that if P is prime number then $X^{P-1}+X^{P-2}+...+X+1$ is irreducible. **[6]**

c) Show that $X^4+10X+5$ is irreducible in $\mathbb{Z}[X]$. Is it irreducible in $\mathbb{Q}[X]$?**[4]**

Q6) a) Show that an abelian group G is $\mathbb{Z}$-module. **[4]**

b) Define the term module homomorphism and give two examples of module homomorphisms. **[4]**

c) If G is finite abelian group then show that G considered as $\mathbb{Z}$-module is not a free $\mathbb{Z}$-module. Is every infinite abelian group a free module? Justify your answer. **[8]**

Q7) a) Find all $\mathbb{Z}$-module homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$. **[4]**

b) Define free module. Give an example of non free module. Justify your answer. **[6]**

c) Let M and N be R-modules, let $\phi : M \to N$ be R-module homomorphism. Show that $\text{Ker}(\phi)$ and $\text{Image}(\phi)$ are R-submodules of M and N respectively. **[6]**

Q8) a) If R is a Boolean ring with unity then show that $x = -x$ for all $x \in R$ and that R is commutative ring. Give an example of Boolean ring. **[6]**

b) Describe all maximal ideals of $\mathbb{R}[X]$. **[4]**

c) Show that in a commutative ring with 1, a prime element is irreducible but not conversely. **[6]**

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[5221] - 33

M.A./M.Sc.

MATHEMATICS

MT - 703 : Mechanics

(2008 Pattern) (Semester - III)

Time : 3 Hours]   *[Max. Marks : 80*

Instructions to the candidates:

1)   Attempt any five questions.

2)   Figures to the right indicate full marks.

Q1) a)   Explain the terms: **[6]**

   i)   Linear momentum

   ii)   Generalized momentum

   iii) Holonomic constraint

   iv)   Rheonomic constraint

b)   A particle is constrained to move on the surface of a cylinder of fixed radius. Find the Lagrange's equation of motion.   **[6]**

c)   Show that the Lagrange's equation of Motion can also be written as **[4]**

$$\frac{\partial L}{\partial t} - \frac{d}{dt}\left(L - \sum \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

Q2) a)   Show that the geodesic in a Euclidean plane is a straight line.   **[5]**

b)   Explain the D'Alembert's Principle and derive Lagrange's equation of motion using the same.   **[7]**

c)   Explain the degree of freedom and find the degree of freedom of a free particle moving in a plane.   **[4]**

Q3) a) Explain Atwood machine and discuss its motion. **[6]**

b) Find E-L differential equation satisfied by twice differentiable function $y(x)$ which extremizes the functional $I(y(x)) = \int_{x_1}^{x_2} f(x, y, y')\, dx$ where y is prescribed at the end points. **[8]**

c) Explain the Basic lemma. **[2]**

Q4) a) Find extremal of the functional $I = \int_{0}^{\frac{\pi}{2}} (y'^2 - y^2 + 2xy)\, dx$, subject to the conditions that $y(0) = 0$, $y\left(\dfrac{\pi}{2}\right) = 0$. **[6]**

b) Find the extremals for an isoperimetric problem **[8]**

$I(y(x)) = \int_{0}^{1} (y'^2 - y^2)\, dx$, subject to the conditions that

$\int_{0}^{\pi} y\, dx = 1,\ y(0) = 0,\ y(\pi) = 1.$

c) Explain the Hamilton's Principle. **[2]**

Q5) a) Describe the motion of a particle of mass m moving near the surface of the Earth under the Earth's constant gravitational field by Hamilton's procedure. **[6]**

b) Prove that a co-ordinated which is cyclic in the Lagrangian is also cyclic in the Hamiltonian. **[4]**

c) Write Hamilton's equations of motion for a compound pendulum. **[6]**

Q6) a) Explain the Principle of least action. **[2]**

b) Prove that central force motion is always motion in a plane. **[6]**

c) Prove the Kepler's third law of planetary motion. **[8]**

Q7) a) Prove that in case of an orthogonal transformation the inverse matrix is identified by the transpose of the matrix. **[8]**

b) Infinitesimal rotation of a right body with one point fixed is commutative. Also find the inverse matrix of infinitesimal rotation. **[8]**

Q8) a) Prove that Poisson brackets are invariant under canonical transformation.**[8]**

b) Show that the transformation defined by $q = \mathrm{P}^2 + \mathrm{Q}^2$, $p = \dfrac{1}{2}\tan^{-1}\left[\dfrac{\mathrm{P}}{\mathrm{Q}}\right]$ are canonical transformation, hence find the generating functions. **[8]**

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[5221] - 34

M.A./M.Sc.

MATHEMATICS

MT-704 : Measure and Integration

(2008 Pattern) (Semester - III) (Old Course)

Time : 3 Hours]   *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*
2) *Figures to the right indicate full marks.*
3) *All symbols have their usual meanings.*

Q1) a) Let $\{E_i\}$ be a sequence of measurable sets. If $E_1 \supseteq E_2 \supseteq ....$, and $m(E_1) < \infty$, then show that $\mu(\lim E_i) = \lim \mu(E_i)$. **[5]**

b) Show that every non-empty open set G in $\mathbb{R}$ is a union of countably many disjoint open intervals. **[5]**

c) Define a σ-algebra. Show that the class of measurable sets $\mathscr{M}$ is a σ-algebra. **[6]**

Q2) a) Let $\{(X_\alpha, \mathscr{B}_\alpha, \mu_\alpha)\}$ be a collection of measurable spaces, and suppose that the sets $\{X_\alpha\}$ are disjoint and define $X = \bigcup X_\alpha$, $\mathscr{B} = \{B : (\alpha)[B \cap X_\alpha \in \mathscr{B}_\alpha]\}$ and $\mu(B) = \sum \mu_\alpha (B \cap X_\alpha)$.

i) Show that $\mathscr{B}$ is a σ-algebra. **[5]**

ii) Show that μ is a measure. **[5]**

b) Define outer measure. Show that the outer measure of an interval equals its length. **[6]**

Q3) a) If *f* and *g* are integrable functions and E is a measurable set then show that

   i) $\int_E (c_1 f + c_2 g) = c_1 \int_E f + c_2 \int_E g$. **[2]**

   ii) If $|h| \le |f|$ and *h* is a measurable then *h* is integrable. **[2]**

   iii) If $f \ge g$ a.e. then $\int f \ge \int g$. **[2]**

b) Show that for any set A and real number $\in > 0$, there is an open set O containing A such that $\mu^*(O) \le \mu^*(A) + \in$. **[5]**

c) Let *f* be defined on [0, 1] by $f(0) = 0, f(x) = x \sin 1/x$ for $x > 0$. Find the measure of the set $[x : f(x) \ge 0]$. **[5]**

Q4) a) Let μ, ν and λ be σ-finite. Show that the Radon-Nikodym derivative $\left[\dfrac{d\nu}{d\mu}\right]$ has the following properties : **[8]**

   i) If $\nu \ll \mu$ and *f* is a nonnegative measurable function, then
   $$\int f \, d\nu = \int f \left[\frac{d\nu}{d\mu}\right] d\mu.$$

   ii) $\left[\dfrac{d(\nu_1 + \nu_2)}{d\mu}\right] = \left[\dfrac{d\nu_1}{d\mu}\right] + \left[\dfrac{d\nu_2}{d\mu}\right]$.

   iii) If $\nu \ll \mu \ll \lambda$ then $\left[\dfrac{d\nu}{d\lambda}\right] = \left[\dfrac{d\nu}{d\mu}\right]\left[\dfrac{d\mu}{d\lambda}\right]$.

b) Let $(X, \mathscr{g}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete measure spaces and *f* an integrable function on $X \times Y$. Then prove the following : **[8]**

   i) For almost all *x* the function f_x defined by $f_x(y) = f(x, y)$ is an integrable function Y.

   ii) For almost all *y* the function f_x defined by $f_y(x) = f(x, y)$ is an integrable function X.

   iii) $\int_Y f(x, y) d\nu(y)$ is an integrable function on X.

   iv) $\int_X f(x, y) d\mu(y)$ is an integrable function on Y.

Q5) a) Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space, and let v be a measure defined on $\mathcal{B}$ which is absolutely continuous with respect to μ. Then prove that there is a nonnegative measurable function f such that for each set E in $\mathcal{B}$ we have $vE = \int_E f d\mu$. **[6]**

b) Let $(X, \mathcal{B})$ be a measurable space and $\langle \mu_n \rangle$ a sequence of measures on $\mathcal{B}$ such that for each $E \in \mathcal{B}$, $\mu_{n+1} E \geq \mu_n$ and $\mu E = \lim \mu_n E$. Then show that μ is a measure on $\mathcal{B}$. **[6]**

c) Show that the Cantor-like set $P^{(\alpha)}$ is measurable with measure $1 - \alpha$. **[4]**

Q6) a) Define product measure. Let E (subset of $X \times Y$) a set in $\mathcal{R}_{\sigma\delta}$ and x be a point of X. Then show that E_x (x cross section E) is a measurable subset of Y. **[6]**

b) Let $\langle f_n \rangle$ be a sequence of measurable functions that converge to a function f except at the points of set E of measure zero. Then prove that f is a measurable function if μ is complete. **[6]**

c) Show that the set of numbers in [0, 1] which possess decimal expansions not containing the digit 5 has measure zero. **[4]**

Q7) a) Suppose that to each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathcal{B}$ such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. Then show that there exist a unique measurable extended real valued function f on X such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on $X \sim B_\alpha$. **[6]**

b) Let $(X, \mathcal{B})$ be a measurable space, $\langle \mu_n \rangle$ a sequence of measures that converge set wise to a measure μ and $\langle f_n \rangle$ a sequence of nonnegative measurable functions that converge pointwise to the function f then show that $\int f d\mu \leq \underline{\lim} \int f_n d\mu_n$. **[6]**

c) If E and F are disjoint sets then show that $\mu_* E + \mu_* F \leq \mu_* (E \bigcup F) \leq \mu_* E + \mu^* F \leq \mu^* (E \bigcup F) \leq \mu^* E + \mu^* F$. **[4]**

Q8) a) Let B be a μ^*-measurable set with $\mu^* B < \infty$ then prove that, $\mu_* B = \mu^* B$. **[4]**

b) Let μ^* be a topologically regular outer measure on X then prove that each Borel set is μ^* - measurable. **[4]**

c) Show that the class $\mathcal{B}$ of μ^* - measurable sets is a σ-algebra. Further, if $\bar{\mu}$ is μ^* restricted to $\mathcal{B}$ then prove that $\bar{\mu}$ is complete measure on $\mathcal{B}$. **[8]**

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M.A./M.Sc.

MATHEMATICS

MT-705: Graph Theory

(2008 Pattern) (Semester - III)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*

2) *Figures to the right indicate full marks.*

Q1) a) Prove that every u, v - walk contains a u, v - path. **[4]**

b) Prove that the Petersen graph has girth 5. **[4]**

c) Define an isomorphism from a simple graph G to a simple graph H with suitable example. Prove that the isomorphism relation is an equivalence relation on the set of simple graphs. **[8]**

Q2) a) Let G be a graph whose vertex set is the set of k - tuples in {0, 1}, with x adjacent to y when x and y differ in exactly one position. Determine whether G is bipartite or not? **[4]**

b) Prove that if P and Q are two paths of maximum length in a connected graph G, then P and Q have a common vertex. **[4]**

c) Prove that the complete graph k_n can be expressed as the union of k bipartite graphs if and only if $n \leq 2^k$. **[8]**

Q3) a) Prove that in a simple graph G, the $(i, j)^{th}$ entry in the kth power of the adjacency matrix A is the number of (v_i, v_j) walks of length k in G. **[4]**

b) Show that the number of vertices in a self - complementary graph is either 4k or 4k + 1, where k is a positive integer. **[4]**

c) Prove that the center of a tree is either a vertex or an edge. **[8]**

*Q4)* a)   State and prove the Havel - Hakimi theorem.   **[8]**

b)   Prove that repeatedly applying the Augmenting path algorithm to a bipartite graph produces a matching and a vertex cover of equal size.   **[8]**

Q5) a)   Show that the hypercube Q_3 has nine perfect matchings.   **[4]**

b)   Prove that if G is a simple graph with diam $G \geq 3$, then diam $\bar{G} \leq 3$. **[4]**

c)   Let $\alpha'(G), \beta'(G)$ and n(G) denotes maximum size of a matching, minimum size of edge cover and number of vertices in a graph G, respectively. Prove that if G is a graph without isolated vertices, then
$$\alpha'(G) + \beta'(G) = n(G).$$ **[8]**

Q6) a)   Using Kruskal's algorithm in the following graph, construct a minimum weight spanning tree.   **[4]**
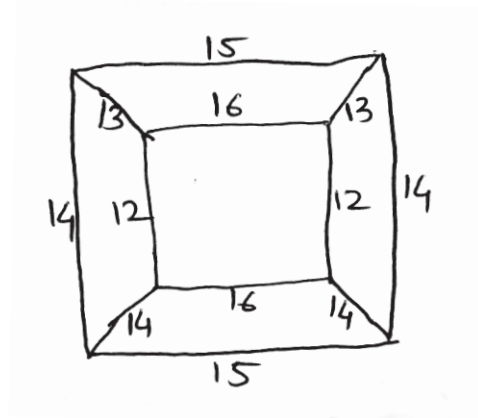
b)   Prove that the connectivity of the hypercube Q_k is k.   **[6]**

c)   Prove that if x and y are distinct vertices of a graph G, then the minimum size of an x, y - disconnecting set edges equals the maximum number of pairwise edge - disjoint x, y - paths.   **[6]**

Q7) a) Prove that the number of spanning trees of a connected graph is the product of the numbers of spanning trees of each of its blocks. **[6]**

b) Prove that a matching M in a graph G is a maximum matching in G if and only if G has no M - augmenting path. **[6]**

c) Prove that if G is a connected graph, then an edge cut F is a bond if and only if G - F has exactly two components. **[4]**

Q8) a) Prove that a graph G is 2 - connected if and only if it has an ear decomposition. **[7]**

b) Prove that a graph G is 2^k colorable if and only if G is the union of k bipartite graphs. **[7]**

c) Let T be a tree with average degree a. Determine n(T) in terms of 'a'. **[2]**

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M.A./M.Sc.

MATHEMATICS

MT - 801:Field Theory

(2008 Pattern) (Semester - IV) (Old)

Time : 3 Hours]   *[Max. Marks : 80*

Instructions to the candidates:

    1)   *Attempt any five questions.*

    2)   *All questions carry equal marks.*

    3)   *Figures to the right indicate full marks.*

Q1) a)   Let $f(x) \in \mathbb{Z}[x]$ be a primitive polynomial. Prove that $f(x)$ is reducible over $\mathbb{Q}$ if and only if $f(x)$ is reducible over $\mathbb{Z}$.   **[6]**

    b)   Show that the polynomial $x^7 - 10x^5 + 15x + 5$ is not solvable by radicals over $\mathbb{Q}$.   **[6]**

    c)   Show that there exists an angle that cannot be trisected by using ruler and compass only.   **[4]**

Q2) a)   Let $f(x) \in \mathbb{Q}[x]$ be a monic irreducible polynomial over $\mathbb{Q}$ of degree p, p is prime. If $f(x)$ has exactly two non real roots in $\mathbb{C}$, then show that the Galois group of $f(x)$ is isomorphic to S_p, where S_p is a symmetric group on a set with p symbols.   **[8]**

    b)   Show that if a real number $x > 0$ is constructible, then $\sqrt{x}$ is also constructible.   **[4]**

    c)   Show that $1 + x + ... + x^{p-1} \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$, where p is prime.   **[4]**

Q3) a)   Let $f(x) = a_0 + a_1 x + ... + a_n x^n \in \mathbb{Z}[x], n \geq 1$. If there is a prime p such that $p^2 \nmid a_0, p \mid a_1, p \mid a_2, ..., p \mid a_{n-1}, p \nmid a_n$, then show that $f(x)$ is irreducible over $\mathbb{Q}$.   **[6]**

b) Show that the Galois group $G(Q(\alpha)/Q)$, where $\alpha^5 = 1$ and $\alpha \neq 1$, is isomorphic to the cyclic group of order 4. **[6]**

c) Find the smallest extension of $\mathbb{Q}$ having a root of $x^4 - 2 \in \mathbb{Q}[x]$. **[4]**

Q4) a) Show that if K is a field of characteristic $p \neq 0$, then K is perfect if and only if $K^p = K$, *i.e.*, if and only if every element of K has p^{th} root in K. **[8]**

b) Let F be a field and $p(x)$ be an irreducible polynomial in $F[x]$. Then show that there exists field extension E of F in which $p(x)$ has a root. **[4]**

c) Let E be an extension field of F. If $a \in E$ has a minimal polynomial of odd degree over F, show that $F(a) = F(a^2)$. **[4]**

Q5) a) Let E be an extension of field F, and let $u \in E$ be algebraic over F. Let $p(x) \in F(x]$ be a polynomial of the least degree such that $p(u) = 0$. Prove that **[6]**

i) $p(x)$ is irreducible over F.

ii) If $g(x) \in F[x]$ is such that $g(u) = 0$, then $p(x) | g(x)$.

b) Show that the degree of extension of the splitting field of $x^3 - 2 \in \mathbb{Q}[x)$ is 6. **[6]**

c) Prove that in a finite field, every element can be written as the sum of two squares. **[4]**

Q6) a) Let E and E' be algebraic closures of a field F. Show that E is isomorphic with E' under an isomorphism that is an identity on F. **[6]**

b) Let E be the splitting field of $x^n - a \in F[x]$. Show that $G(E/F)$, the Galois group, is solvable. **[6]**

c) Let E be the splitting field of a polynomial of degree n over a field F. Show that $[E:F] \leq n!$. **[4]**

Q7) a) Prove that any irreducible polynomial $f(x)$ over a field of characteristic 0 has simple roots. Also show that any irreducible polynomial $f(x)$ over a field F of characteristic $p \neq 0$ has multiple roots and only if there exists $g(x) \in F(x]$ such that $f(x) = g(x^p)$. **[8]**

b) Define simple extension. Show that every finite separable extension of a field F is a simple extension. **[4]**

c) Show that the field generated by a root of $x^3 - x - 1$ over $\mathbb{Q}$ is not normal over $\mathbb{Q}$. **[4]**

Q8) a) Show that every finite field F with p^n elements is the splitting field of $x^{p^n} - x \in F_p[x]$, where F_p is the subfield of F with p elements. Also show that any two finite fields with p^n elements are isomorphic. **[8]**

b) Show that a finite field F of p^n elements has exactly one subfield with p^m elements for each divisor m of n. **[4]**

c) Show that if $f(x) \in F[x]$ is irreducible over F, then all roots of $f(x)$ have the same multiplicity. **[4]**

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M.A./M.Sc.

MATHEMATICS

MT-802 : Combinatorics

(2008 Pattern) (Semester - IV)

Time : 3 Hours]   *[Max. Marks : 80*

Instructions to the candidates:

   1) Attempt any five questions.

   2) Figures to the right indicate full marks.

Q1) a) How many arrangements of the letters in 'MISSISSIPPI' in which **[6]**

     i) The M is immediately followed by an I?

     ii) The M is beside an I, that is, an I is just before or just after the M.

  b) Prove by combinatorial argument that $C(r,r) + C(r+1,r) + C(r+2,r) + \ldots\ldots\ldots + C(n,r) = C(n+1,r+1)$. Hence evaluate the sum $1^2 + 2^2 + \ldots\ldots\ldots + n^2$. **[6]**

  c) Find two different chessboard(not row or column rearrangements of one another) that have the same rook polynomial. Also, write the rook polynomial. **[4]**

Q2) a) What fraction of all arrangements of 'INSTRUCTOR' have three consecutive vowels? **[6]**

  b) How many arrangements are there of 'TAMELY' with either T before A, or A before M or M before E? [By before, we mean anywhere before, not just immediately before). **[6]**

  c) How many nonnegative integer solutions are there to the pair of equations $x_1 + x_2 + \ldots\ldots\ldots + x_6 = 20$ and $x_1 + x_2 + x_3 = 7$? **[4]**

Q3) a) How many ways are there to place an order for 12 chocolate sundaes if there are 5 types of sundaes and at most 4 sundaes of one type are allowed? **[6]**

b) Find ordinary generating function whose coefficient a_r equals $3r^2$. Hence, evaluate the sum $0 + 3 + 12 + \ldots\ldots\ldots + 3n^2$. **[6]**

c) How many distributions of 24 different objects into three different boxes are there with twice as many objects in one box as in the other two combined? **[4]**

Q4) a) Using generating function, find the number of distributions of 18 chocolate bunny rabbits into four Easter baskets with atleast 3 rabbits in each basket. **[6]**

b) How many ways are there to collect $24 from 4 children and 6 adults if each person gives atleast $1, but each child can give at most $4 and each adult at most $7? **[6]**

c) Show that in any set of n integers, $n \geq 3$, there always exists a pair of integers whose difference is divisible by n–1. **[4]**

Q5) a) How many n-digit decimal sequences (using digits $0,1,2\ldots\ldots\ldots9$) are there in which digits 1,2,3 all appear? **[6]**

b) How many ways are there to make an r-arrangements of pennies, nickels, dimes and quarters with at least one penny and an odd number of quarters? **[6]**

c) Solve the recurrence relation $a_n = 2a_{n/2} + 2, n \geq 4$, with $a_2 = 1$. **[4]**

Q6) a) Uisng generating functions, solve the recurrence relation,
$a_n = 2a_{n-1} + 2^n$ with $a_0 = 1$ **[6]**

b) How many ways are there to assign 20 different people to three different rooms with atleast one person in each rooms? **[6]**

c) Solve the recurrence relation **[4]**
$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$, with $a_0 = a_1 = 1; a_2 = 2$

Q7) a) Find recurrence relation for the number of n-digit binary sequences with an even number of zeros. **[6]**

b) Suppose a school with 120 students offers Yoga and Karate. If the number of students taking Yoga alone is twice the number taking Karate(Possibly, Karate and Yoga), if 25 more students study neither skill than study both skills, and if 75 students take at least one skill, then how many students study Yoga? **[6]**

c) How many arrangements of the letters in 'MATHEMATICS' are there in which TH appear together but the TH is not immediately followed by an E? **[4]**

Q8) a) Five officials $O_1, O_2, \ldots, O_5$ are to be assigned five different city cars $C_1, C_2, \ldots, C_5$. If O_1 will not drive C_1 or C_5; O_2 will not drive C_2 or C_3; O_3 will not drive C_3; O_4 will not drive C_1 or C_5; O_5 will not drive C_3. How many ways are there to assign the officials to different cars? **[8]**

b) Solve the recurrence relation $a_n^2 = 2a_{n-1}^2 + 1$ with $a_0 = 1$. **[4]**

c) Show that if n+1 distinct numbers are chosen from 1,2,.......2n, then two of the numbers must always be consecutive integers. **[4]**

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[5221]-43

M.A/M.Sc.

MATHEMATICS

MT - 803 : Differential Manifolds

(2008 Pattern) (Semester - IV)

Time : 3 Hours]   *[Max. Marks : 80*

Instructions to the candidates:

1) *Attempt any five questions.*

2) *Figures to the right indicate full marks.*

Q1) a)   What is the dimension of $A^K(V)$, the space of alternating k-tensors on an n-dimensional vector space V? Justify.   **[8]**

b)   State Green's theorem for compact oriented 2 - manifold.   **[4]**

c)   Define exact form and give example.   **[4]**

Q2) a)   Define orientation of a manifold M and induced orientation on ∂M.   **[4]**

b)   State Stokes' theorem.   **[4]**

c)   Let $\alpha : (0,1)^2 \to \mathbb{R}^3$ be given by $\alpha(u,v) = (u,v,u^2+v^2+1)$. Let Y be the image set of α. Evaluate

$$\int_Y x_2 dx_2 \wedge dx_3 + x_1 x_3 dx_1 \wedge dx_3.$$ **[8]**

Q3) a)   Define a closed form and give example.   **[4]**

b)   Let V be an open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ be of class C^r. Let $M = \{x : f(x) = 0\}$ and $N = \{x : f(x) \geq 0\}$. If M is non - empty and D$f(x)$ has rank one at each point of M, then prove that N is an n-manifold in $\mathbb{R}^n$ and $\partial N = M$.   **[8]**

c) Give example of a manifold which can be covered by a single coordinate patch. **[4]**

Q4) a) Define the differential operator d and for any k-form w show that $d(dw) = 0$. **[7]**

b) Show that $g(X,Y,Z) = \det \begin{pmatrix} x_i & y_i & z_i \\ x_j & y_j & z_j \\ x_k & y_k & z_k \end{pmatrix}$ is an alternating 3-tensor on $\mathbb{R}^n$.

Further express g as a combination of elementary tensors. **[6]**

c) Define volume of parametrized surface in $\mathbb{R}^n$. **[3]**

Q5) a) Let F be a k-tensor. With usual notation, if $AF = \sum_{\sigma \in S_k} (Sign\,\sigma)F^\sigma$, then prove that AF is an alternating tensor Find AF if F is already alternating. **[7]**

b) If $w = x^2 dx + ydy + ze^x dz$ and $n = y\cos x dx + xdy + 2xydz$, then find $(w \wedge n)$. **[5]**

c) Define alternating tensor and give example. **[4]**

Q6) a) Let M be k - manifold in $\mathbb{R}^n$. If ∂M is non - empty, then prove that ∂M is k-1 manifold without boundary. **[7]**

b) If $w = x^2 z^2 dx + 2z\cos ydy + e^z dz$, find dw. **[5]**

c) Find the tangent plane to the unit sphere at (1, 0, 0). **[4]**

Q7) a) If w and n are k and l forms respectively then prove that $d(w \wedge n) = dw \wedge n + (-1)^k w \wedge dn.$. **[8]**

b)   Let $w = y^2 z dx + x^2 z dy + x^2 y dz$ and $\alpha(u,v)=(u-v,uv,u^2)$. Find $\alpha^*(dw)$.

[8]

Q8) a)   If $T:V \rightarrow W$ is a linear transformation and if f and g are alternating tensors on w then prove that $T^*(f \wedge g)=T^*f \wedge T^*g$.   [8]

b)   Let $A = \mathbb{R}^2 - \{0\}$. If $w = \dfrac{xdx + ydy}{x^2 + y^2}$ then show that w is closed and exact on A.   [8]

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M.A./M.Sc.

MATHEMATICS

MT-804: Algebraic Topology

(2008 Pattern) (Semester - IV)

Time : 3 Hours]   *[Max. Marks :80*

Instructions to the candidates:

   1)   *Attempt any five questions.*

   2)   *Figures to the right indicate full marks.*

Q1) a) Define homotopy relation between two continuous functions. Show that this relation is an eqivalence relation. **[6]**

   b) Let $f, g : X \to S^n$ be continuous mappings such that $f(x) \neq -g(x)$ for all $x \in X$. Show that f is homotopic to g. **[5]**

   c) Let f and g be homotopic mappings of X into Y and h be a continuous mapping of Y into Z. Show that hf and hg are homotopic. **[5]**

Q2) a) Define a contractible space. Give an example of a non - contractible space. Show that if Y is contractible and $f : X \to Y$ is a continuous, then f is homotopic to a constant. **[6]**

   b) Define a strong deformation retract. Show that S^n is a strong deformation retract of $\mathbb{R}^{n+1} - 0$. **[5]**

   c) Let X be connected and has the same homotopy type as Y. Show that Y is connected. **[5]**

Q3) a) Prove that a non - empty open connected subset of $\mathbb{R}^n$ is path connected. **[6]**

b) Show that if f is any path then $f * \bar{f}$ and $\bar{f} * f$ are null homotopic. **[5]**

c) Let $A \subset X$ be path connected and let A_n be a path connected subset which intersects A for $n = 1, 2, \ldots$ Show that $A \cup \left(\bigcup_{n=1}^{\infty} A_n \right)$ is path connected. **[5]**

Q4) a) Show that the set $\pi_1(X, x_0)$ of homotopy classes of paths in X at $x_0 \in X$ is a group. **[6]**

b) Define a simply connected space. Show that a contracible space is simply connected but the converse is not true. **[5]**

c) Let X be path connected and $x_0, x_1 \in X$. Prove that the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic. **[5]**

Q5) a) Prove that the fundametal group of S^1 is isomorphic to the additive group $\mathbb{Z}$. **[6]**

b) Prove that the fundamental group of the real projective plane is a cyclic group of order two. **[5]**

c) Prove that the circle is not a retract of the disc B^2. **[5]**

Q6) a) Let $p : E \rightarrow B$ be a fibration such that every fiber has no non - null path. Prove that p has unique path lifting property. **[6]**

b) Determine the fundamental groups of $\mathbb{R}^2 - 0$ and $\mathbb{R}^n$. **[5]**

c) Find the fundamental group of torus. **[5]**

Q7) a) Define a covering map. Prove that a covering map is a local homeomorphism but the converse is not true. **[6]**

b) Prove that $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is a covering map. **[5]**

c) Show that a retract of a Hausdorff space is a closed subset. **[5]**

Q8) a) Prove that the diameter of a simplex is the length of its longest edge. **[6]**

b) Using algebraic graph theory techniques, prove that the closed ball has the fixed point property. **[5]**

c) Prove that every complex has a barycentric subdivision. **[5]**

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M. A./M.Sc.

MATHEMATICS

MT - 805: Lattice Theory

(2008 Pattern) (Semester - IV)

Time : 3 Hours] *[Max. Marks :80*

Instructions to the candidates:

1) *Attempt any five questions.*

2) *Figures to the right indicate full marks.*

Q1) a) Let the poset $<L, \leq>$ be a lattice. Set $a \wedge b = \inf\{a,b\}$ and $a \vee b = \sup\{a,b\}$. Then prove that the algebra $<L; \wedge, \vee>$ ia a lattice. **[5]**

b) Let L be a lattice and let I be nonempty subset of L. Prove that I is an ideal if and only if the following two conditions hold: **[6]**

i) $a, b \in I$ implies that $a \vee b \in I$,

ii) $a \in I$, $x \in L$, and $x \leq a$ imply that $x \in I$.

c) Let L, L_1, K, K_1 be lattices, $L \cong L_1$, $K \cong K_1$. Prove that $L \times K \cong L_1 \times K_1$. **[5]**

Q2) a) Let A be a set of all subgroups of a group G. Prove that A is a lattice under set inclusion. **[6]**

b) Define Isotone map and show that every homomorphism is isotone map. Is the converse true? Justify your answer. **[6]**

c) If a poset satisfies Descending Chain Condition then prove that it has minimal element. **[4]**

Q3) a) Let I be an ideal and let D be dual ideal if $I \cap D \neq \phi$, then prove that $I \cap D$ is a convex sublattice and every convex sublattice can be expressed in this form in one and only one way. **[7]**

b) Prove that I is a proper ideal of L if and only if there is a join homomorphism ϕ of L onto C_2 with $I = \phi^{-1}(0)$. **[5]**

c) Prove that every element of a finite distributive lattice has a unique irredundant representation as a join of join irreducible elements. **[4]**

Q4) a) Prove that every ideal of L is prime if and only if L is chain. **[6]**

b) If P be a poset, $\wedge H$ exists for all $H \subset P$ then prove that P is complete lattice. **[6]**

c) Draw Hasse diagram of divisors of 20 with $a \leq b$ if and only if a divides b.**[4]**

Q5) a) In a bounded distributive lattice if an element has complement then prove that it has relative complement in any interval containing it. **[5]**

b) Prove that a collection of normal subgroup of any group is modular lattice. **[6]**

c) Prove that every ideal I of a distributive lattice is the intersection of all prime ideals containing it. **[5]**

Q6) a) Prove that a lattice is Boolean if and only if it is isomorphic to field of sets.**[6]**

b) Let L be distributive lattice, $a, b \in$ L and a $\neq$ b, then prove that there exist a prim ideal P of L containing exactly one of a and b. **[6]**

c) Let θ be the congruence relation on L, then prove that for every $a \in$ L, $[a]\theta$ is convex sublattice of L. **[4]**

Q7) a) Prove that a lattice L is distributive if and only if it does not contain a sublattice isomorphic to M_3 or N_5. **[8]**

b) State and prove Jordan-Hölder Theorem for semimodular lattices. **[8]**

Q8) a) Prove that every modular lattice is semimodular. Is the converse true? Justify your answer. **[6]**

b) Let L be a bounded distributive lattice with $0 \neq 1$. Then prove that L is a Boolean lattice if and only if P(L) is unordered. **[6]**

c) Prove that every distributive lattice is modular. **[4]**

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